Euclidean Geometry

In this topic learners should be able to:

- Investigate, conjecture and prove theorems of the geometry of circles assuming results from earlier grades and accepting that the tangent to a circle is perpendicular to the radius drawn to the point of contact.
- Solve circle geometry problems and prove riders, using circle geometry theorems, their converses and corollaries (where they exist) as well as geometry results establish in earlier grades.

Content:

a. Circles, perpendicular lines through the centre, chords and midpoints; Angle subtended at the centre of a circle; Angle subtended by diameter; Cyclic quadrilaterals; angents to circles
b. Solutions of geometric riders and problems

Learning outcomes:

Learners should be able to:

3.2.1 Investigate, conjecture, prove the following theorems of the geometry of circles:

a. The line drawn from the centre of the circle perpendicular to the chord bisects the chord.
b. The perpendicular bisector of a chord passes through the centre of the circle.
c. The angle subtended by an arc at the centre of the circle is double the size of the angle subtended by the same arc at the circle (on the same side as of the chord as the centre).
d. The angle subtended at the circle by a diameter is a right angle.
e. Angles subtended by a chord of the circle on the same side of the chord, are equal.
f. The opposite angles of a cyclic quadrilateral are supplementary.
g. An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.
h. Two tangents drawn to a circle from the same point outside the circle are equal in length.
i. The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment. (tan-chord theorem)

Note: Accept results established in earlier grades as axioms and also that a tangent to a circle is perpendicular to the radius, drawn to the point of contact.

Range: The proof of the following theorems will be examined

a. The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.
b. The angle subtended by an arc at the centre of the circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre).
c. The opposite angles of a cyclic quadrilateral are supplementary.
d. The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment

3.2.2 Use the above theorems, their converses and corollaries (where they exist) to solve and/prove riders:

a. Theorems from section 3.2.1
b. Equal chords subtend equal angles at the circumference
c. Equal chords subtend equal angles at the centre
d. Equal chords of equal circles subtend equal angles at the circumference.
e. If the angle subtended by a chord at point on the circle is a right angle, then the chord is a diameter.
f. If a line is drawn perpendicular to a radius at the point where the radius meets the circle, then it is a tangent to the circle.
g. If the exterior angle of a quadrilateral is equal to the interior opposite angle then the quadrilateral will be a cyclic quadrilateral.
h. If two opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.
i. If through one end of a chord of a circle, a chord is drawn making with the chord an angle equal to the angle in the alternate segment, then the line is a tangent to the circle.

NB: Provisionally, 2 Units have been developed to cover the geometry topic as geometry is a bug-bear for learners. We call them UNIT 8 and UNIT 9 respectively. For now, I have been writing the material and have not focused too much on the technical layout. I will still have to edit some mathematical symbols using the equation editor. I will also use the proper angle symbol later when I am editing]

Unit 8: Using inductive reasoning to study circles

Inductive reasoning - is a kind of reasoning in which a conclusion is based upon experimentation and observation of patterns in several specific cases. It is one of the methods used for discovering geometric relationships in a circle.

[nsert photograph of a circular watch and a bicycle with caption]

The circle is one of the most familiar geometric figures. Many everyday objects like bicycle wheels are based on circles. Circles have been important figures throughout history. The ancient Greeks believed that the sun, planets, and other celestial objects went around the earth in circles. The greatest scientist Galileo Galilei wrote in 1623:

….the universe … is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering in a dark labyrinth.

You have had many experiences with circle and circular objects. This chapter introduces you to properties of circles. Using your geometric tools, you will discover many relationships among the angles and line segments around circles. In particular we will focus on the following aspects:

- The line drawn from centre of a circle perpendicular to a chord.
- The line drawn from the centre of a circle to the midpoint of a chord.
- The relationship between the angle subtended by an arc at the centre and at the circle.
- Angles subtended by a chord at the circle on the same side of the chord.
- The opposite angles of a cyclic quadrilateral.
- Two tangents drawn to a circle from the same point outside the circle.
- Angle between a tangent and a chord.

UNIT 8.1 Circles and Chords

Investigation 8A : Some definitions to get started

A **definition** gives the meaning of a word in terms of other words you already understand. It should be stated as **economically** as possible and should fit **usefully** into our deductive chain. In grade 11 you studied definitions of similar figures. Now you can study definitions in the context of circles.

We begin with the definition of a circle:

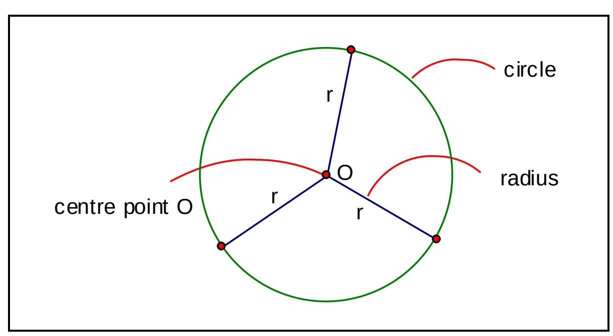
A **circle** is the set of all points in a plane that are a given distance (r) from a fixed point in that plane.

The fixed point (O) is the **center** of the circle.

The given distance (r) is the **radius** of the circle.

A segment from the center to any point on the circle is a r**adius**(plural: **radii**

A circle is named by its center. The circle alongside is



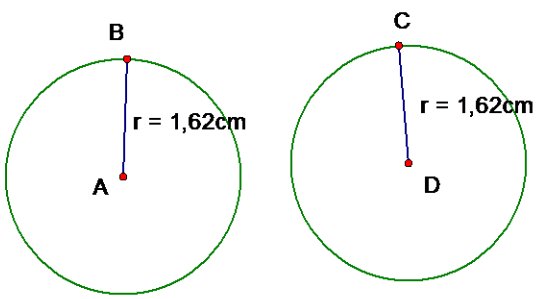
named Circle O or ©O ,where the symbol for circle is ©.

Now you can write your own definitions:

1. Study the examples and non examples of the given geometric terms. Write a definition for each geometric term.
2. Discuss your definitions with others in your class. Agree on a common set of definitions and then add them to your definition list.
3. In your workbook, draw and label a picture to illustrate each definition.

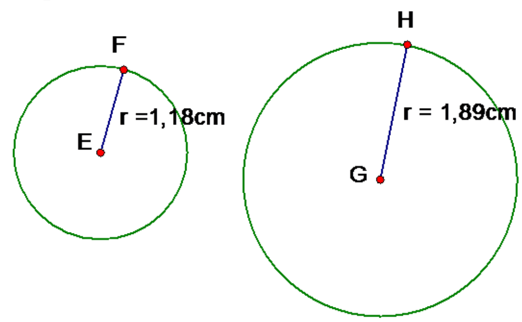
(a) Define **congruent** circles

Congruent circles



©A with radius AB = 1,62 cm and
©D with radius CD = 1,62 cm and
congruent circles.

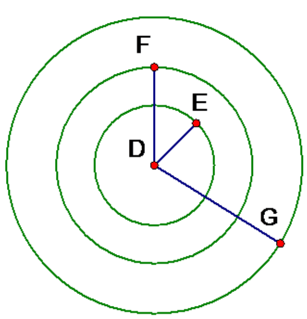
not congruent circles



©E with radius EF = 1,18 cm and
©G with radius GH = 1,89cm are
not congruent circles.

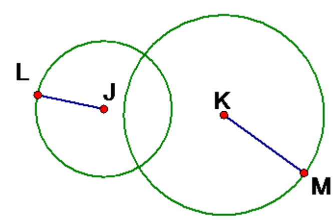
(b) Define **concentric** circles

concentric circles



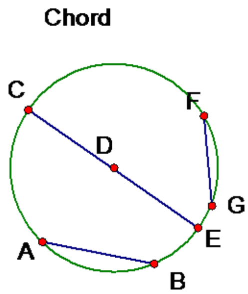
©D with radius DE, ©D with radius DF
and ©D with radius DG are concentric
circles.

not concentric circles

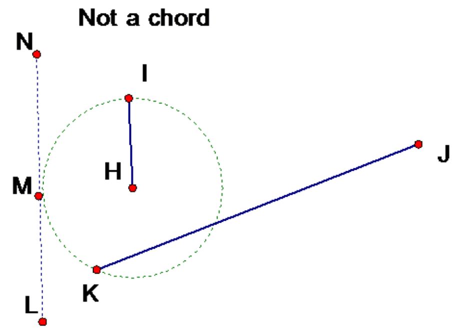


©J with radius LJ and ©K with
and ©D with radius KM are not
concentric circles.

(c) Define **chord**

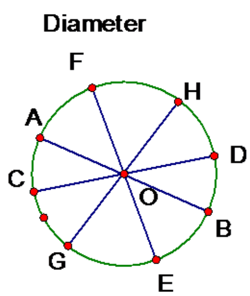


AB, CE and FG are chords of $\odot D$

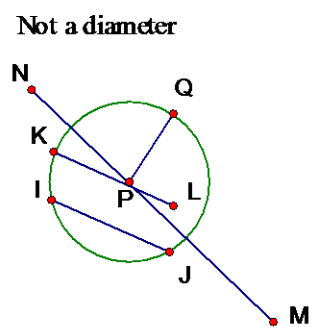


HI, JK and LMN are not chords of $\odot H$

(d) Define **diameter**



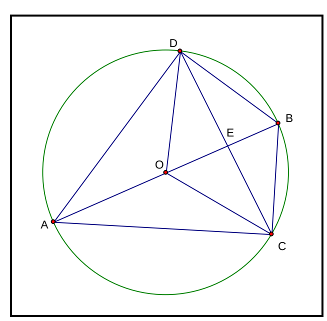
AB, CD, EF and GH are diameters of $\odot O$



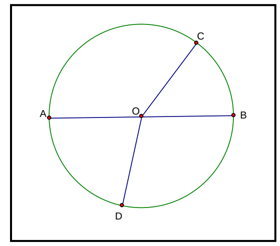
IJ, KP, KL ,PQ and NM are not diameters of $\odot H$

Exercise 8.1

1. 1. Use the figure on the right to identify :
   (a) three chords
   (b) one diameter
   (c) four radii



2. (a) Use your compass to draw a circle.
   (b) Draw several diameters and radii.
   (c) Measure the diameters and radii. Compare their lengths.
   (d) Write down your finding(s).



3. Use your finding(s) in question 2(d) to answer the following question.

   In ©O , AB = 8cm.
   Determine the length of OC and OD.

4. The figure at the right shows a Vodacom tower that sends out signals that reaches 25 km.

   (a)What is the maximum distance two people can live from each other and still receive that station's signal.
   (b) If Inderani's cell phone receives the station's signal, what can be said about her distance from the station.
   (c) What is the distance from the northernmost point of the station's signal the southernmost point?
   (c) Trevelyan and Shannon are each 25 km from the Vodacom Tower and also 25km from each other. Draw a picture showing how that can be.

5. Draw two circles, ©A and ©B, each with a radius of 5 cm and intersecting at two points C and D. What type of quadrilateral is ABCD?
6. Draw two circles, © A and © B, each with a diameter of 6 cm , so that each circle passes through the center of the other. Label the points of intersection C and D. What is the length of [line segment] AB?
7. Explain why a diameter is the longest chord of a circle.
8. (a) How many circles can be drawn through three points in a plane? When is it impossible to draw such a circle?
   (b) Plot three points A,B,C on your page. How would you find the centre of the circle, which would go through these points.
   (c) Construct the circle.
9. How many circles can be drawn through two fixed points A and B in a plane?
10. How many circles of radius 6cmcan be drawn through two fixed points A and B in a plane?

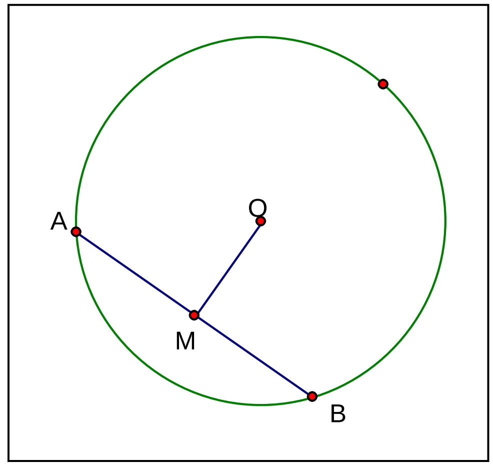
UNIT 8.2 : More about Chords in a Circle LO3 AS 3.2 (a), 3.2 (b)

INVESTIGATION 8B : Investigating chords in a circle
Here is a definition of a chord: A chord in a circle is a segment with endpoints on the circle.

Compare this definition to your definition of a chord in INVESTIGATION 8A. Do you think this new definition is a good definition?

You will need a ruler and compass in this Investigation. Use computer software if possible.

1.



Step 1 Draw a circle. Mark the center O.

Step 2 Draw any chord AB other than a diameter.

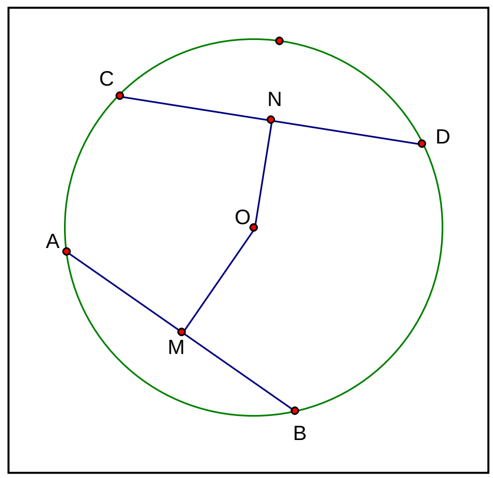
Step 3 Draw a line from the center of the circle perpendicular to chord AB. Use the letter M to indicate the point of intersection.

Step 4 Measure the length of AM and MB. What do you notice?

Compare your results with the results of others near you. State your observations as a conjecture. Remember, a conjecture is a mathematical statement that we think is correct, but is not yet proved.

Conjecture 1: The line drawn from the center of the circle, perpendicular to a chord _________ the chord.

2.



Step 1 Draw a circle. Mark the center O.

Step 2 Draw any chord AB other than a diameter.

Step 3 Determine the midpoint of chord AB and label it using the letter M.

Step 4 Draw a line segment joining the center of ©O and the midpoint M of chord AB.

Step 5 Measure ∠OMA. What can you say about OM?

Step 6 Draw another chord CD other than a diameter in the same © O.

Step 7 Determine the midpoint of chord CD and label it using the letter N.

Step 8 Draw a line segment joining the center of ©O and the midpoint N of chord CD.

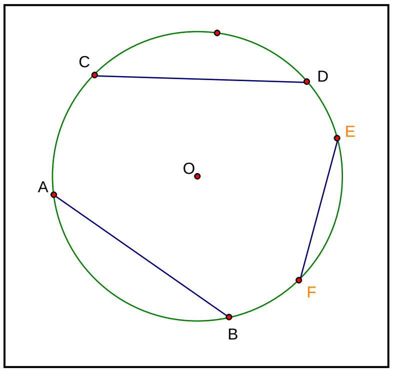
Step 9 Measure ∠ONA. What can you say about ON?

Compare your results in steps 5 and 9, with the results of others near you. State your observations as a conjecture.

Conjecture 2: The line segment drawn from the center of the circle to the midpoint of a chord is _________ to the chord.

3.

(a) Draw a circle. Mark the centre O.
(b) Draw chords AB, CD and EF.
(c) Construct the perpendicular bisector of each chord.
(d) What do you observe about the perpendicular bisector of each chord.
(e) Compare your results with results of your peers and formulate a conjecture.



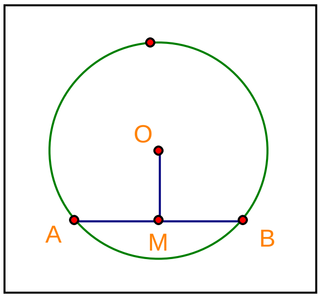
Conjecture 3: The perpendicular bisector of a chord passes through the _______.

FOR DISCUSSION

Use your investigations to check whether you agree with the following results. Remember that you have used lots if circles and inductive reasoning to arrive at these results. In chapter 9 you will use deductive reasoning to prove theses results for all circles.

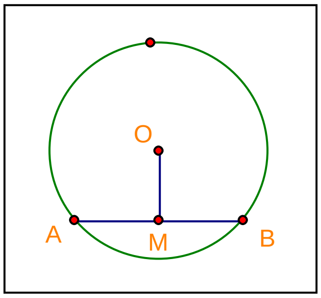
1. **CONJECTURE 1**

Given : Circle O with OM perpendicular to AB
Conclusion: AM = MB
Reason: Perpendicular form centre to chord.



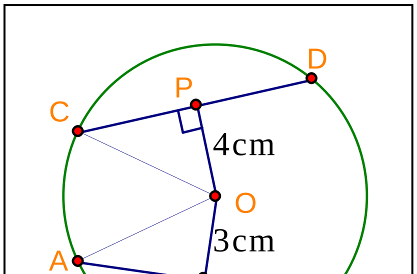
2. **CONJECTURE 2**

Given : Circle O with M the midpoint of chord AB, i.e. AM = BM
Conclusion: OM perpendicular to AB
Reason: Line through centre and midpoint.



In the following worked example you can see how we used these two results to solve a geometry problem. Look at how we explain our reasoning in the brackets next to each statement.

In circle O, AB = 8 cm, OM = 3cm,



AM =BM, OP ⊥ CD and OP = 4cm.
Calculate : (a) The radius of the circle.
(b) the length of CD

SOLUTION:
Construction: Join AO and CO
Calculation:

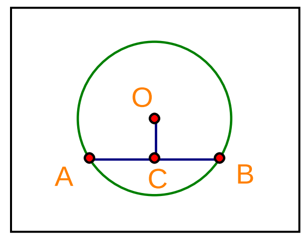
(a) AM =MB = 4cm ….(AM =BM, given)
<OMA = 90º …(M midpoint of chord AB)
$\therefore$radius OA = 5 cm ….(Pythagoras triple 3; 4 and5 in Δ AOM)

(b) CO = AO = 5cm …(radii)
$\therefore$CP = 3cm …(Pythagoras triple 3; 4 and5 in Δ COP)

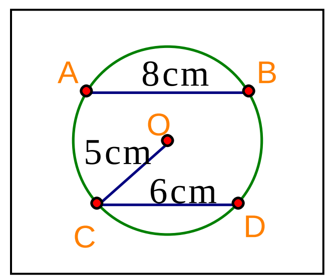
CD = 2CP … (OP ⊥ chord CD)
$\therefore$CD = 6cm

Exercise 8.2

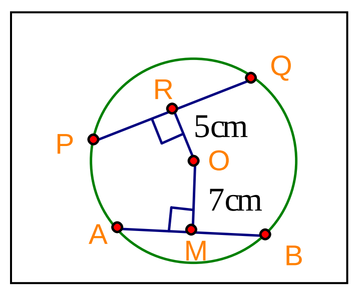
1.In ©O, C is a midpoint of chord AB.
(a) If radius AO = 5cm, AB = 8cm, find OC.
(b) If radius AO = 13cm, OC = cm, find AB.
(c) If OC = 8cm, AC = 15cm, find radius AO..



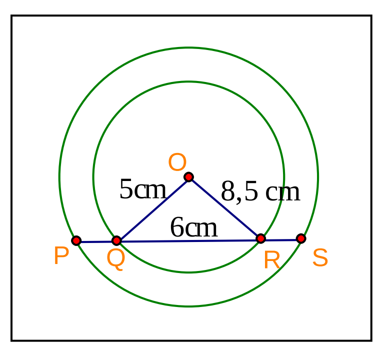
2. In ©O, chord AB = 8 cm , chord CD = 6cm, radius OC = 5cm and AB// CD. Calculate the distance between AB and CD.



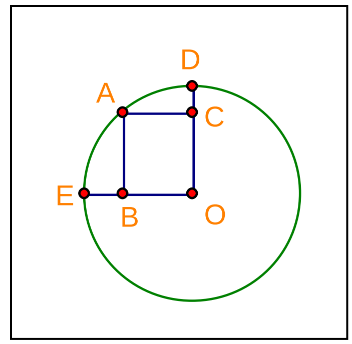
3. In ©O, chord AB = 48 cm , OM ⊥ chord AB, OR ⊥ chord PQ, OM = 7cm and OR = 5cm. Calculate the length of PQ.



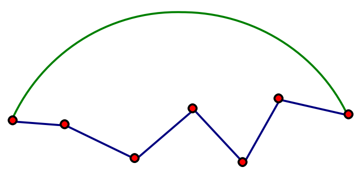
4. Two concentric circles O have radii 5 cm and 8,5 cm respectively. Calculate the length of PS.



5. In ©O, ACOB is a rectangle. BC = 26 mm , Dc = 16 mm.
Find:
(a) the radius of ©O.
(b) the area of rectangle ACOB.



6.



The circular glass top of your neighbor's coffee table breaks. Your neighbor is very upset and would like to replace the glass top but does not know how big it was. He brings you a piece of broken glass that contains part of the boundary of the original top. Describe exactly what you would do in order to figure out the six of the original glass table top.

7. A large object recently crashed into the lunar surface, leaving a gigantic circular crater(between marks 24, 25,34, and 35). Some scientists have reason to believe the object may have been launched by creatures from another galaxy. Unfortunately, the single photo received from the satellite shows only a portion of the new crater. Cartographers must locate centre of the crater and determine its radius in order to program the lunar land vehicle to retrieve the mysterious space object. Trace the outer edge of the crater shown on the photo onto a sheet of paper. Locate the crater's centre. Using the scale shown, determine the radius. Express your answer in kilometers. [INSERT PHOTO]

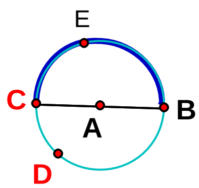
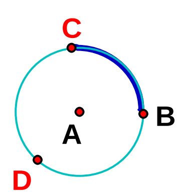
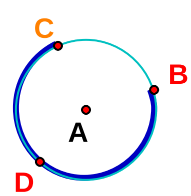
UNIT 8.3: Angles and Arcs

[Insert a picture showing the use of angles and arcs. Also give a brief to the picture]

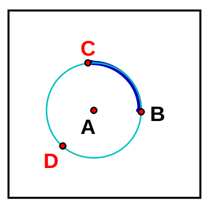
Discussion

Before we can study more properties of a circle, we need some more definitions.
An arc of a circle consists of two points on the circle and the continuous (unbroken) part of the circle between the two points. The two points are called the endpoints of the arc.

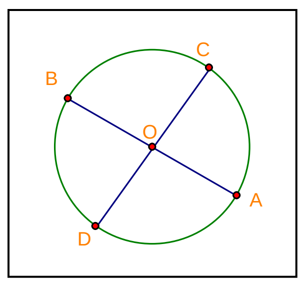
Some special types of arcs and their notation are defined and illustrated below:

Semicircle	**Minor Arc**	**Major Arc**
A semicircle is an arc of a circle whose endpoints are the end points of a diameter.	A minor arc is an arc of a circle that is less than a semicircle of the circle.	A major arc is an arc of a circle that is greater than a semicircle of the circle
Three letters are used to name a semicircle. The first and the last letters are the endpoints and the middle letter is any other point on the arc.	A minor arc may be named with either two or three letters. We will name a minor arc with the letters of the two endpoints of the arc.	To avoid confusion, we will always use three letters to name a major arc. The first and the last letters are the endpoints and the middle letter is any other point on the arc.
[line segment symbol] CB is a diameter [arc symbol] CEB and [arc symbol] CDB are semi circles of ©A. 	Minor arc[arc symbol] CB ©A. 	Major arc [arc symbol]CDB of ©A. 

Special note:
Points B and C divides the circle into two arcs.
The shorter arc (which is the thick arc in the figure alongside)
is called the *minor arc* and the longer one is called the *major arc*.
A minor arc is named after its endpoints.
In the figure alongside [arc] BC is the minor arc.



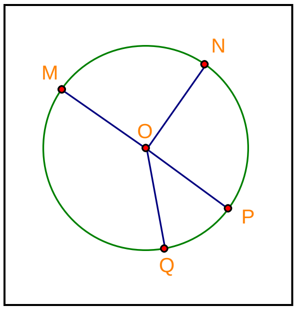
Example:
In the diagram on the right , name the minor arcs,
major arcs and semi circles.
Solution:
Minor arcs: [arc]AB, [arc]BC, [arc]CD and [arc] DA.
Major arcs: [arc]ADB (or[arc]ACB), [arc]BAC (or[arc]BDC),



[arc]DCA (or[arc]DBA) and
[arc]DAC (or[arc]DBC)

Semicircles: [arc]DAB, [arc]DCB, [arc]ABC, and [arc]ADC.

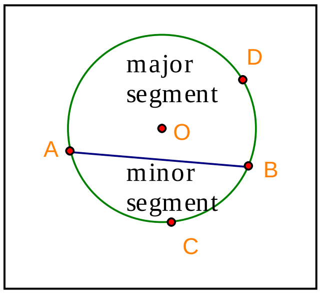
Try this:
MP is the diameter of © O.
Name the minor and major arcs and semicircles.



For Discussion

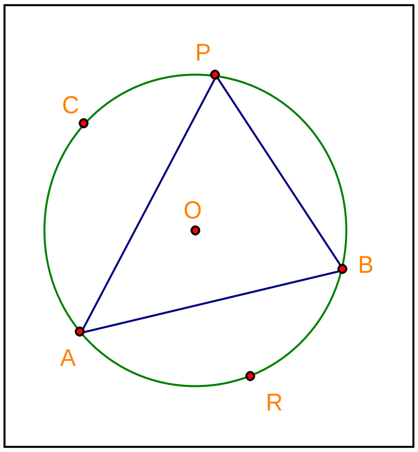
1. Segments of a circle

A segment of a circle is the area enclosed between a chord an arc.



ACB is a **minor segment.**
ADB is a **major segment.**

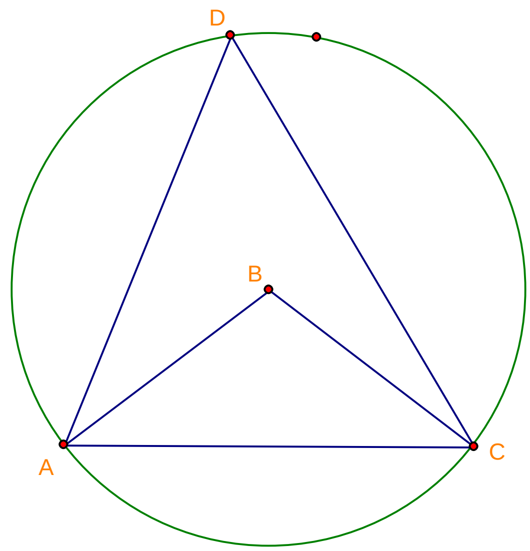
2. **An angle in a segment of a circle**
∠APB is an angle in the segment ACB because its vertex, P, is on the arc ACB and its arms meet the circumference of the circle at A and B.



We say that the chord AB subtends ∠APB or that the arc ARB subtends ∠APB, at the circumference (or circle).

∠APB can also be described as an angle at the circumference of the circle subtended by minor arc AB.

3. Angles subtended by an arc (or chord) of a circle



An **angle at the centre** of a circle is an angle whose vertex is at the centre of a circle and whose sides contains radii of the circle.
[Angle] ABC is an angle at the centre of circle B

An **angle at the circle** (circumference) is an angle whose vertex lies on a circle and whose sides contain chords of the circle.

[angle] ADC is an angle at the circle (circumference).

We say that minor arc AC (or chord AC) **subtends** [non-reflex angle] ABC at the centre and [angle] ADC at the circle (circumference), which lies in the major segement ADC.[Name the non-reflex angle ABC as alpha or B1and the reflex angle beta or B2].

In the diagram alongside, state the angle that the the majpr arc ADC subtends at the centre.

Copy the adjacent figure into your book, and draw an angle subtended by the major arc ADC at the circle.

INVESTIGATION 8D: Angle at centre and angles in the same segment

INVESTIGATION 8D (a): Angle at centre

1. In this activity you will investigate the relationship between an angle at the centre and the angle at the circle.

Copy this table. Complete the table as you go through steps 1 to 16.

Case	Measure of angle at center	Measure of angle at circle	How does the measure of angle at center compare with the measure of angle at circle
Diagram 1	< AOC =	< ABC =	
Daigram 2	< AOC =	< ABC =	
Diagram 3	< AOC =	< ABC =	
Diagram 4	Reflex < AOC =	< ABC =	
Diagram 5 (your own)			

In diagrams 1-4, < AOC is the angle at center of ©] O and < ABC is the angle at the circle (or circumference) of ©O.

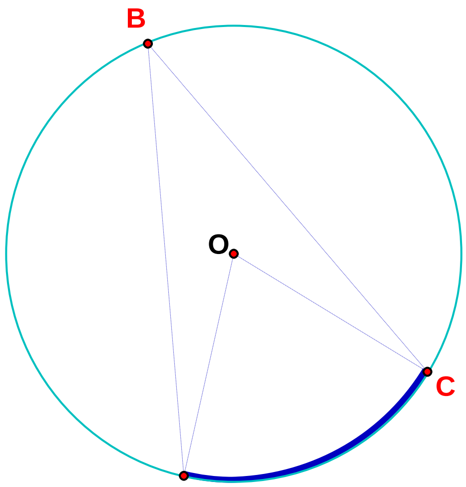


Diagram 1

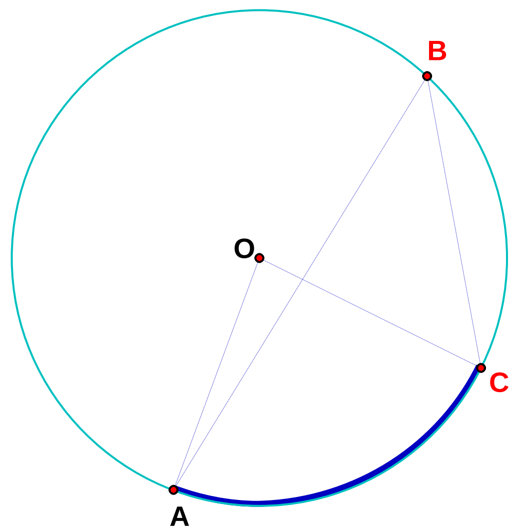


Diagram 2

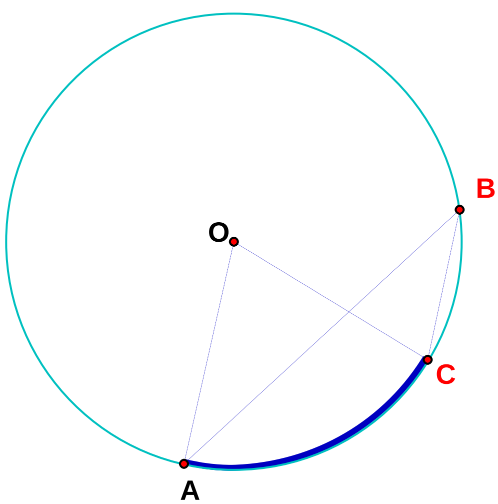


Diagram 3

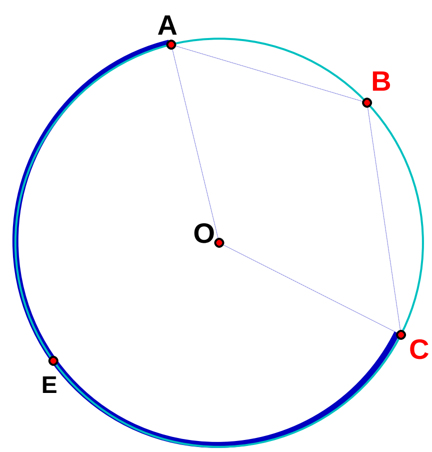


Diagram 4

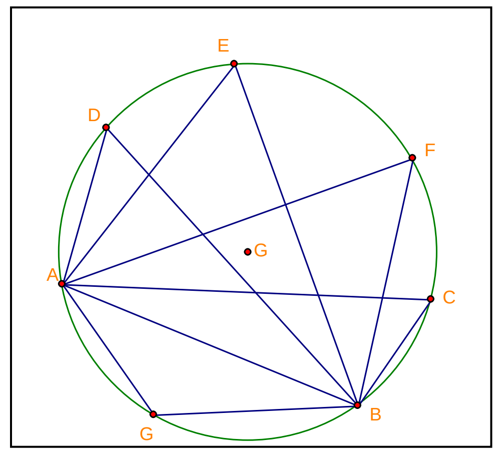
Step 1. Use a protractor to measure < AOC in diagram 1.
Step 2. Measure < ABC in diagram 1.
Step 3. In diagram 1, how does the measure of < AOC compare with the measure of < ABC.
Step 4. Use a protractor to measure < < AOC in diagram 2.
Step 5. Measure < ABC in diagram 2.
Step 6. In diagram 2, how does the measure of < < AOC compare with the measure of < ABC.
Step 7. Use a protractor to measure < < AOC in diagram 3.
Step 8. Measure < ABC in diagram 3.
Step 9. In diagram 3, how does the measure of < < AOC compare with the measure of < ABC.
Step 10. Use a protractor to measure reflex < < AOC in diagram 4.
Step 11. Measure < ABC in diagram 4.
Step 12. In diagram 4, how does the measure of < AOC compare with the measure of < ABC.

Step 13. Construct a circle of your own with an arc subtending and angle at center and an angle at the circle (or circumference) . Call this diagram 5.
Step 14. Measure the angle at center in diagram 5.
Step 15. Measure the angle at the circle (or circumference) in diagram 5. How does the measure of angle at center compare with the measure of angle at circle.
Step 16. Study the table and write a conjecture about the measure of the angle subtended by arc at center of a circle and the angle subtended by the same arc at the circle.
What kind of reasoning did you use.

Conjecture 4: The measure of the angle subtended by arc at center of a circle is ________
(**Angle at Centre Conjecture**)

INVESTIGATION 8D (b): Angles in the same segment

In ©G on the right, chord AB subtends angles C, D, E , F and G at the circle .



(a). Without measuring, make a conjecture about the measures of <C,<D,<E, <F and <H

(b). (a) Now use a protractor to measure <C,<D,<E and <F.
Use a table like this record your answers:

Measure of < C	Measure of < D	Measure of < E	Measure of < F	Measure of < G

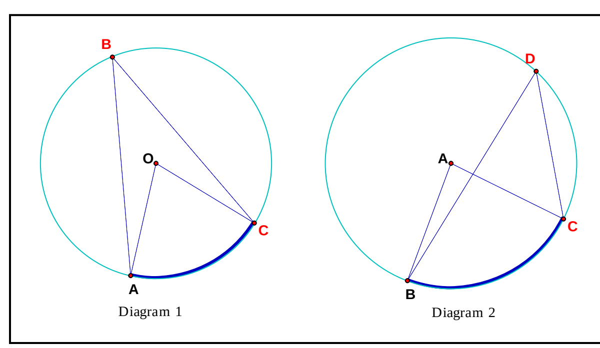
(b) What appears to be true about <C,<D,<E and <F.
(c) Measure <H. Fill in your measurement in the table.
(d) Why do you think the measure of <H is different from the measure of <C or <D or<E or<F?
(e) Study your table and make a generalization. What kind of reasoning did you use?

Conjecture 5: Angles subtended by a chord at the circle on the same side of the chord are ______

For Discussion

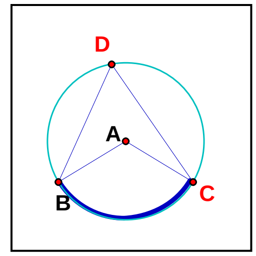
In this section we summarise some of the properties of the circle. Make sure these results agree with your investigations. You will use deductive reasoning to prove these results in Chapter 9.

1. **Conjecture 4:**
Given: Circle with centre at O and arc BC subtending <AOC at the centre and <ABC at the circle.
Conclusion: < AOC = 2< ABC.
Reason: Angle at centre = 2X < on circumference (or at circle)



Note: We that we say that a line “subtends” an angle when that line is opposite to the angle i.e. The arms of the angle pass through the extremities of that line.

2. **Example 1**:
In the diagram below, A is the centre of the circle.
If $<BAC = 122°$, determine the size of $<D$.



SOLUTION:

RTC: $<D$

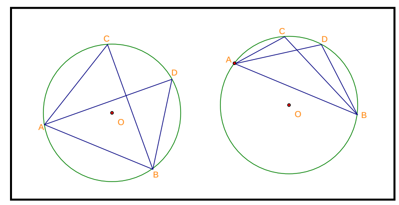
Calc.: $<BAC = 2 <D$…(Angle at centre = 2X < at circle)

But $<BAC = 122°$ … (given)

Therefore $2 <D = 122°$

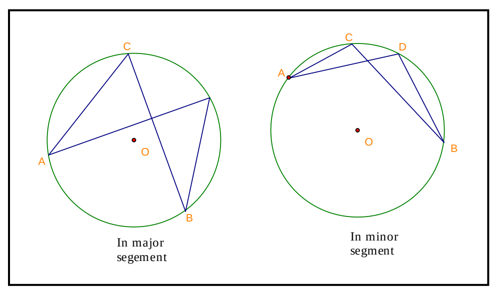
$<D = 61°$

3. **Conjecture 5:**
Given: Circle O and chord AB subtending $<C$ and $<D$ on the same side of chord AB.
Conclusion: $<C = <D$.
Reason: <’s subtended on the same side of the chord.

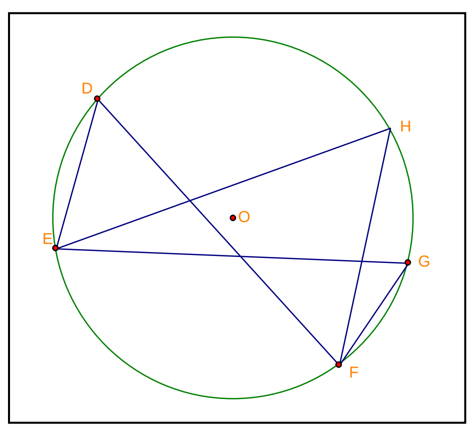


4. Conjecture 5, may also be worded thus:
The angles in the same segment of a circle
(**Angle in the same segment Conjecture**)

Given: $<C$ and $<D$ on segment ACDB of circle O.
Conclusion: $<C = <D$.
Reason: <’s in same segment



5. **Example 2:**
In the diagram below, O is the centre of the circle and $< D = 54°$. Determine the sizes of $<H$ and $<G$.



Solution:

RTC: $<H$ and $<G$

Calc.: $<D = <H$ (<’s in the same segment)

but $< D = 54°$

$\therefore$ $< H = 54°$

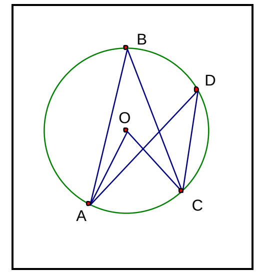
$<D = <G$ (<’s in the same segment)

but $< D = 54°$

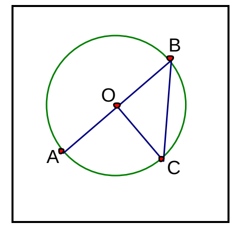
$\therefore < G = 54°$

Exercise 8.3
In this Exercise make sure that you explain your reasoning. Use examples in the Discussion to help you with the setting out.

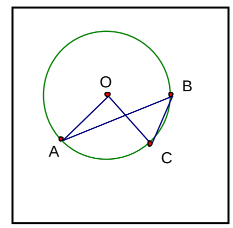
1. In ©O, ∠AOC is an angle at the centre of the circle.
   (a) If ∠AOC = 54° , what is the size of ∠ABC and ∠ADC
   (b) If ∠ABC = 43°, determine the size ∠ADC and the the size of ∠AOC.
   (c) If ∠ADC = 28°, determine the size ∠ABC and the the size of reflex ∠AOC.



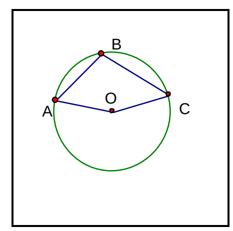
2. Does the relationship between ∠AOC (at centre) and ∠ABC (at circumference) still hold in this sketch?
   Explore



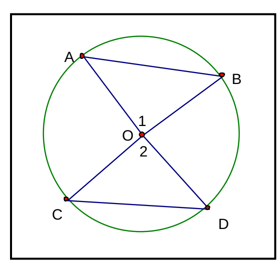
3.Does the relationship between ∠AOC (at centre) and ∠ABC (at circumference) still hold in this sketch?
Explore.



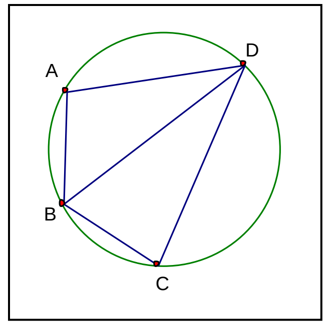
4.Does the relationship between refex ∠AOC (at centre) and ∠ABC (at circumference) still hold in this sketch?
Explore.



5. In the accompanying figure, chord AB = chord CD.
   What can you deduce about <AOB and <COD
   State words what you find.



6. In the accompanying figure, <ADB = <CDB.
What can you deduce about chord AB and chord BC.
State words what you find.



7.

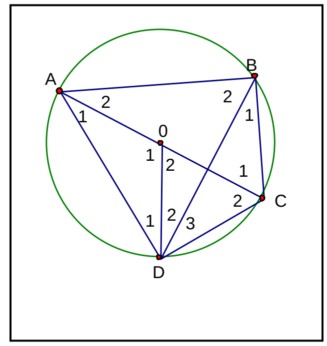
In © O on the right, O is joined to A, D and C .

AOC is a diameter , AD = AB and $\angle O_2 = 44^\circ$. Calculate

(a) $\angle A_1$ (b) $\angle D_1$ c) $\angle O_1$

(d) $\angle B_1$ (e) $\angle B_2$ (f) $\angle A_2$

(g) $\angle C_1$ (h) $\angle D_2$



Discussion

In grades 10 and 11 you learnt the term converse of a statement. Talk about what this term means: To formulate a converse of a statement, the condition and the conclusion should be reversed. Some converses are true statements , others are false.

For look at conjecture below:

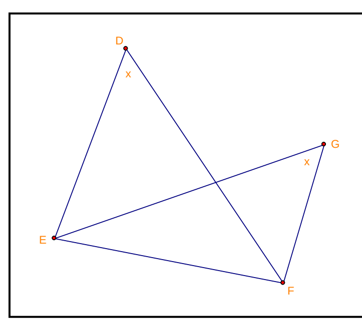
Condition	**conclusion**
Angles subtended by a chord at the circle on the same side of the chord	**are equal**

If we swop the condition and conclusion we have the following statement:

Condition	**conclusion**
If a line segment joining two points subtends equal angles at two other points on the same side of that line segment	then the four points lie on a circle (i.e. they are concylic.

Below you can see an example of this converse:

Given: Points E and F, with EF being the line segment joining them.
Two other points G and H, on the same side of EF,
such that <H = <G.



Conclusion: E,F, G and H are concyclic points
(i.e a circle can be drawn and pass
through E,F, G and H)

Reason: line segment subtends equal angles on the same side

7. We may now write conjecture 6, which is converse of Conjecture 5 as follows:
Conjecture 6: If a line segment joining two points subtends equal angles at two other points on the same sides of that line segment, then the four points are concyclic (i.e. they lie on a circle).
Do you think that conjecture 6 holds? Can you explain why?.

UNIT 8.4:Cyclic Quadrilaterals

Investigation 8E: Some properties of cyclic quadrilateral

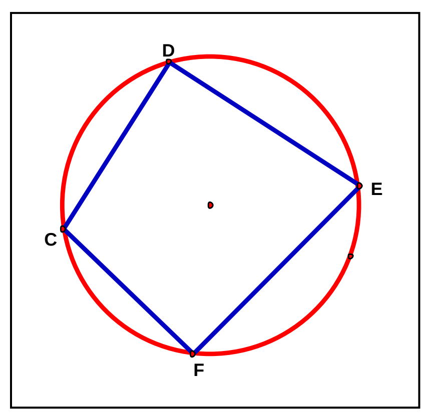
In Grade 10 you studied quadrilaterals. Remember a quadrilateral is ……….
Now we are going to look at the relationships between circles and quadrilaterals.
We begin with definition of a term that you will need:

A quadrilateral is call cyclic quadrilateral if all the vertices of the quadrilateral lie on the circumference of the circle.

[Picture showing the use of cyclic quadrilaterals]
[Brief introduction to picture]

INVESTIGATION 8E (a): Opposite angles of a cyclic quadrilateral

In this activity you will investigate some properties of a quadrilateral inscribed in a circle – in other words, a quadrilateral whose vertices lie on a circle. Such a quadrilateral is called a cyclic quadrilateral.



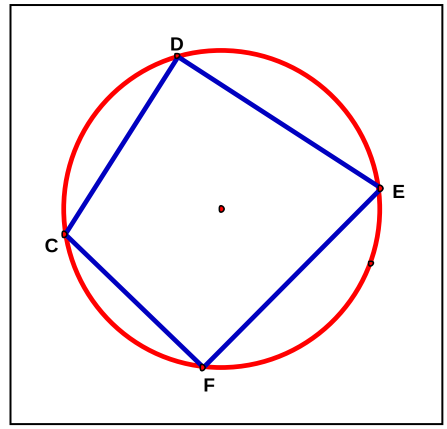
1. Use your protractor to measure each of the following angles in the accompanying figure: <C; <D; <E and <F.
2. Calculate the sum of each pair of opposite angles:
   <D + <F = _________

   <E + <F = __________
3. What can you say about the two pairs of opposite angles of a cyclic quadrilateral.

4. Compare your observations with the observations of those in your group.
5. State your findings as your next conjecture.

Conjecture 7: The _____ angles of a cyclic quadrilateral are _______.

IINVESTIGATION 8E(b): Perpendicular bisectors of the sides of a cyclic quadrilateral



1. Construct a large circle.
2. Construct cyclic quadrilateral CDEF.
3. Construct the perpendicular bisectors of the sides of cyclic quadrilateral CDEF.
4. What do you notice about the perpendicular bisectors of a cyclic quadrilateral CDEF?
5. Compare your observations with the observations of those near you.
6. State your findings as your next conjecture.

   Conjecture 8: The perpendicular bisectors of the sides of a cyclic quadrilateral always ______________________.

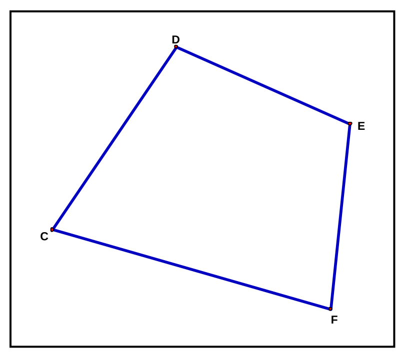
7. Explain why each perpendicular bisector goes through the center of the circle.

Investigation 8E (c) : Opposite angles of a cyclic quadrilateral – converse

A cyclic quadrilateral is any quadrilateral that can be circumscribed by a circle.

In Investigation G with cyclic quadrilaterals, you have observed that if a quadrilateral is cyclic, its opposite angles ________ .

In this activity, you will investigate the converse of this statement. Before you go on, write the converse in your own words.



In the above convex quadrilateral, $<D= 100^{o}$, $< E = 108^{o}$, $< F = 108^{o}$ and $< C = 72^{o}$.

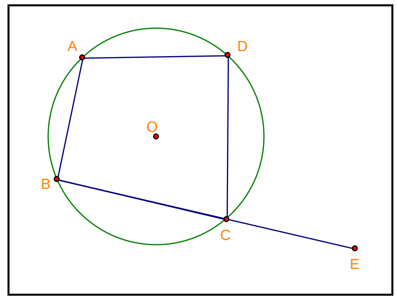
1. What can you say about each pair of opposite angles of quadrilateral CDEF.
2. Construct the perpendicular bisectors of the sides of quadrilateral CDEF.
3. Mark the point of intersection of all the perpendicular bisectors of the sides of quadrilateral CDEF, using the letter O.
4. Measure and record length of OF.
5. Place the needle of your compass at point O. Use the length of OF as your radius and draw a circle.
6. What do you observe about the vertices of quadrilateral CDEF?(or what can you say about the circle O ?).Compare your observation with others in your group or near you.
7. Is quadrilateral CDEF a cyclic quadrilateral or not. Give a reason for your answer.
8. Does your finding(s) supports or contradicts the converse statement you wrote

at the beginning of this activity.

Compare your results with the results of your peers and formulate a conjecture.

Conjecture 9: If the opposite angles of a quadrilateral are supplementary, that quadrilateral is ________.

INVESTIAGTION 8E (d): Exterior angle of a cyclic quadrilateral



In the figure alongside, ABCD is a cyclic quadrilateral. If we extend BC to E, then <DCE is called an exterior angle of cyclic quad.ABCD.

Copy this chart. Complete this chart as you answer Questions 1-6.

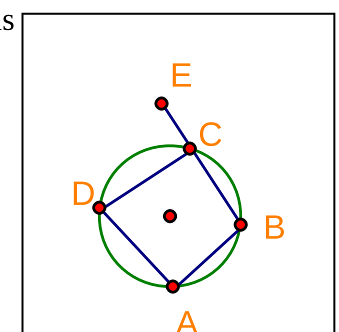
	Measure of < A	Measure of < B	Measure of < BCE	Measure of < D	Measure of < DCE
Diagram 1					
Diagram 2					
Diagram 3					

1. Use a protractor to measure <A, B, < BCE, <D, and <DCE in diagram 1
   Record your readings in the above table.
2. Draw another cyclic quadrilateral ABCD in your workbook. Extend BC to E. Then <DCE is called an exterior angle of cyclic quad.ABCD. You will consider this drawing as diagram 2.
3. Use a protractor to measure <A, B, < BCE, <D, and <DCE in diagram 2
   Record your readings in the above table.
4. Draw another cyclic quadrilateral ABCD in your workbook. Extend BC to E. Then <DCE is called an exterior angle of cyclic quad.ABCD. You will consider this drawing as diagram 3.
5. Use a protractor to measure <A, B, < BCE, <D, and <DCE in diagram 3
   Record your readings in the above table.

6. Study the chart and make a conjecture about the exterior angle of a cyclic quadrilateral. What kind of reasoning did you use?

Conjecture 10: The exterior angle of a cyclic quadrilateral is

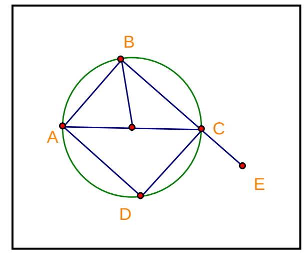
FOR DISCUSSION:



1. **Conjecture 10:**
Given: Cyclic quad ABCD with BC produced to E,
creating ext. < DCE
Conclusion: <A = < DCE
Reason: Exterior angle of cyclic quad.

2. Two worked examples which follow show you how to set your reasons.

(a) Worked example 1



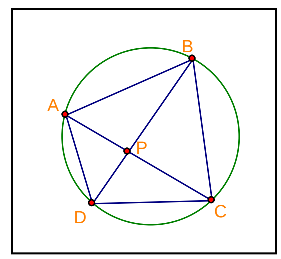
O is the centre of the circle, ∠ABD = 112˚ and AD = DC.
Calculate ∠DCE.

Solution:

∠BAC = 56˚…. (½ ∠BOC at centre)
∠ADC = 90˚…. (∠ in semi ©)
∴ ∠DAC = 45˚…. (AD =DC)
∴ ∠DCE = 56˚+ 45˚…. (ext. ∠ of cyclic quad ABCD)
= 101˚

Remember: Fill in your sketch as you go along.

(b) **Worked example 2**

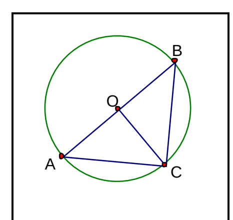


In the diagram on the right, ABCD is a cyclic quadrilateral.
∠ABD = 32˚ and ∠DBC = 56˚ .
Calculate: (a) ∠ACD
(b) ∠ADC

Solution:

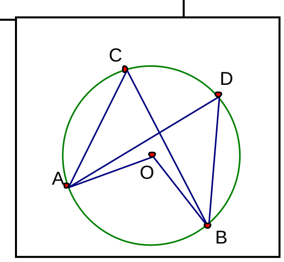
(a) ∠ACD = 32˚ … (∠ in the same segment as ∠ABD)
(b) ∠ADC = 180˚ - 88˚ …(opp. to ∠ABC in cyclic quad ABCD)
= 92˚

Exercise 8.4 LO3



1. In © O on the right, ∠AOC = 82˚. Calculate

(a) ∠B (b) ∠ACB (c) ∠A

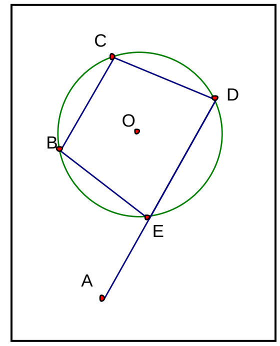


2. In © O on the right, ∠ACB = 74˚ and ∠CAD = 32˚. Calculate

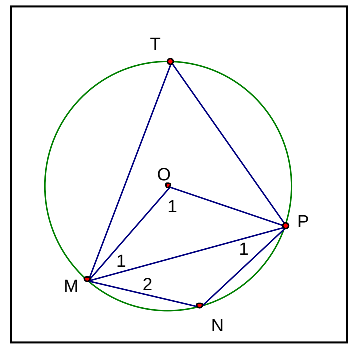
(a) $\angle D$ (b) $\angle AOB$ (c) $\angle CBD$

3. In © O on the right, $\angle BCD = 98^\circ$ and BC//DE. Calculate

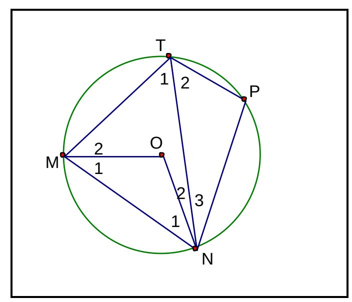
(a) $\angle D$ (b) B (c) $\angle AEB$



4. In © O on the right, O is joined to M and P and $\angle OPM = 46^\circ$. Calculate
(a) $\angle O_1$ (b) $\angle T$ c) $\angle N$



5. In the figure on the right, MNPT is a cyclic quadrilateral.
O is the centre of the circle. OM bisects $\angle TMN$ and OMN = 32°.
(a) Name two other angles equal to 32°.
(b) Determine the size of $\angle NPT$.
(c) Calculate the size of $\angle T_1$



UNIT 8.5 : Tangents and secants

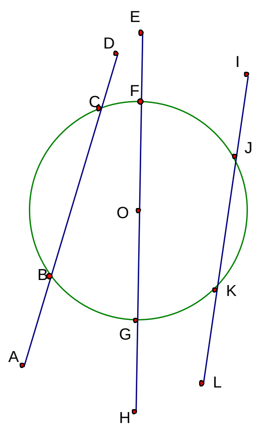
[Insert a picture showing the use of tangents. Also give a brief to the picture]

INVESTIGATION 8F: Definition of tangent and secant
Study the examples and non examples of the given geometric terms. Write

a definition for each geometric term. Discuss your definitions with others in your class. Agree on a common set of definitions and then add them to your definition list. In your workbook, draw and label a picture to illustrate each definition.

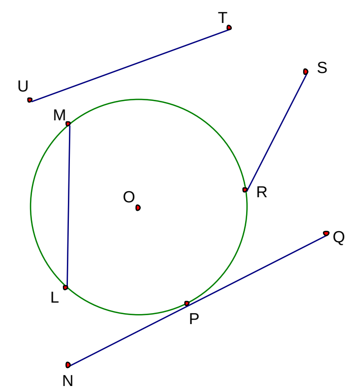
1. Define secant

Secant



ABCD , EFGH and IJKL are secants to circle O.

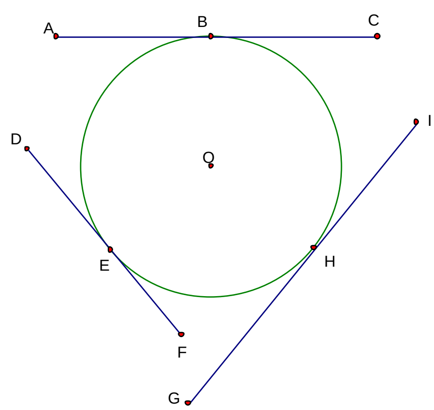
Not a secant



ML, UT, RS and NPQ are not secants to circle O.

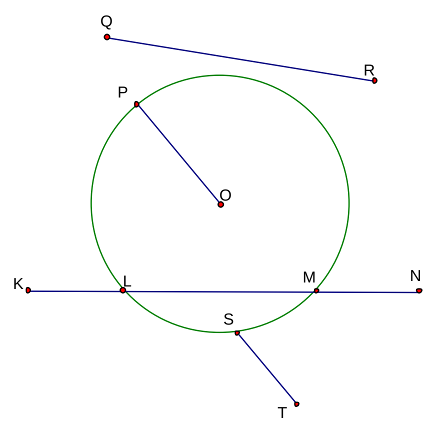
2. Define a tangent

Tangent



ABC, DEF and GHI are tangents

Not a tangent



KLMN, ST, OP and QR are not tangents

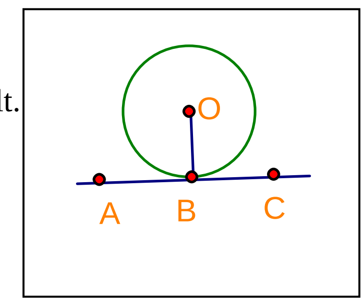
INVESTIGATION 8G: Angle between tangent and radius
In this investigation, you will discover something about the angle formed by a tangent and the radius drawn to the point of tangency.

1. Construct a large circle. Label the center 0.
2. Using your ruler, draw a line that appears to touch the circle at only one point.
3. Label the point T. Point T in this case is called the point of **tangency**.
4. Construct OT.
5. Use your protractor to measure the angles at T.

Compare your results with others near you. State your observations as a conjecture.

Conjecture 10: A tangent to a circle is _____to a radius drawn to the point of tangency (Tangent- Radius Conjecture)

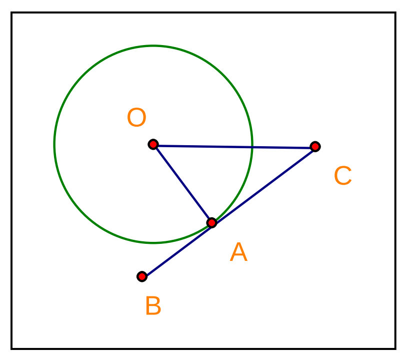
For Discussion



Use your investigation to check whether you agree with the following result. Remember that you have used lots of examples of tangents to circles and inductive reasoning to arrive at this result.

Given: Circle O with ABC tangent at B and OB the radius.
Conclusion: ABC is perpendicular to OB
Reason: tangent perpendicular to radius (or tangent-radius conjecture)

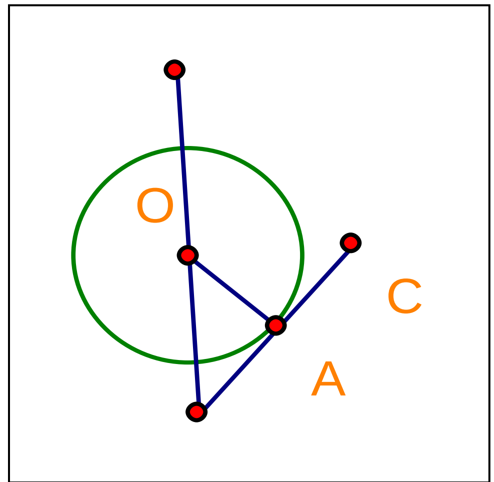
1. Example 1:
BC is a tangent to circle O at A. OA is 6 cm and OC = 10cm. Determine the length of AC.



AC is perpendicular to OA …tangent [perpendicular to] radius)
Therefore <OAC = 90°.

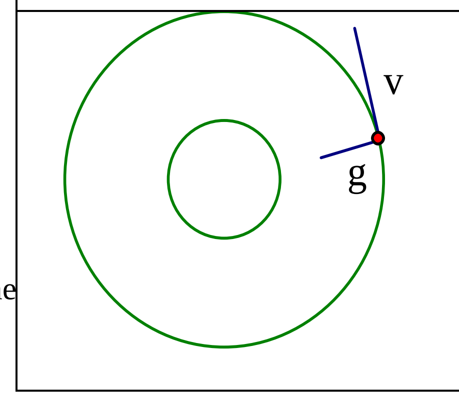
$(6cm)^2 +\ OC^2 = (10cm)^2$ ……………………(Pythagoras theorem)
$\therefore$ $OC^2 = (10cm)^2 - (6cm)^2$
$OC^2 = 100\ cm^2 - 36\ cm^2$
$OC^2 =\ 64\ cm^2$,
and $OC = 8$ cm

Try this



(a) If BA is a tangent to circle O at A, OA = 5cm and AB= 12 cm. Determine the length of OB.
(b) If <DOA = 119°, then determine the size of <B.

Bigger Picture

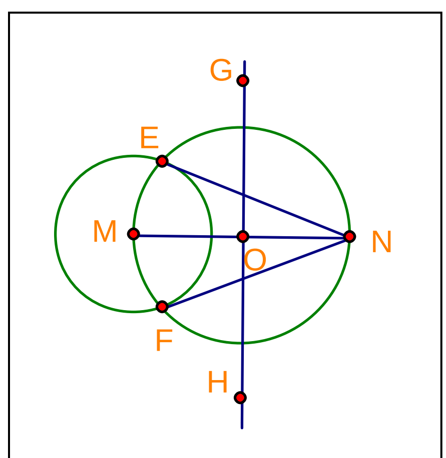


1. The Tangent –Radius Conjecture comes up in many applications related to circular motion. A satellite, for example, maintains its velocity in a direction tangent to its circular orbit. This velocity vector is perpendicular to the force of gravity, which keeps the satellite in orbit. If the gravity were suddenly "turned off" somehow, the satellite would travel off into space on a straight line tangent to its orbit.

2. Find a real-world example that illustrates the tangent –radius conjecture. Either sketch the examples or make photocopies from a book or magazine to put into your book.

INVESTIGATION 8H: Two tangents from a common point

In this activity, you'll learn how to construct two tangents to a circle from the same point outside the circle. Then you will compare the lengths of two tangents from the same point to the points of tangency.



1. Draw circle M.
2. Select a point outside circle M. Label this point N.
3. Draw MN.
4. Construct GH, the perpendicular bisector of MN.
5. Let O be the point of intersection of GH and MN.
6. Using OM as the radius, draw circle 0.
7. Let E and F be the points of intersection of circle O and circle M.
8. Draw NE and NF. Then NE and NF are tangents to

circle M from point N.

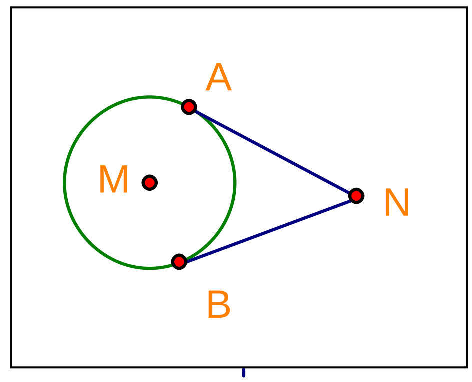
9. Points E and F are points of tangency of tangents NE and NF respectively to circle M.
10. Measure NE and NF [or use your compass to compare tangents NE and NF].
11. Write down a conjecture about tangents drawn from the same point outside the circle to respective the points of tangency.

Conjecture 11: Two tangents drawn to a circle from the same point outside the circle are ________.

Discussion

1. Use your investigation to check whether you agree with the following result. In Chapter 9 you will use deductive reasoning to prove this result for all circles.

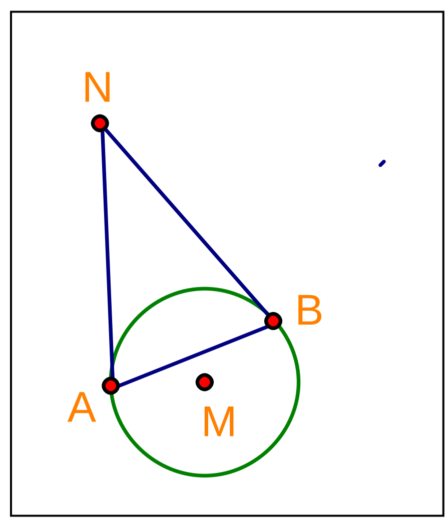
Conjecture 11:
Given: Tangents NA and NB drawn from point N to circle M.
Conclusion: NA =NB
*Reason***:** Tangents drawn from a common point.



2. Example 1:

NA and NB are tangents to circle M. <A = 68°.
Determine the size of <N.

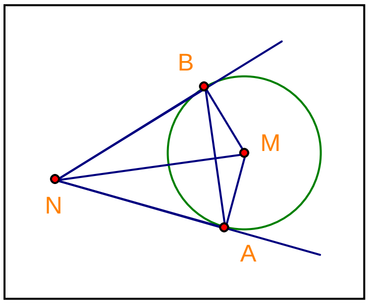
Required to Calculate [RTC]: **<N**
Calculation**:**
NA =NB ……….(Tangents drawn from common point)
Hence, [triangle] NAB is isosceles and <A =<B.
Therefore <A = <B = 68°,
and <N = 180° - (68° + 68°) …sum of angles of a triangle)
= 44°



Exercise 8.5 LO3

1. Draw a circle with a radius of 6cm and select any point outside the circle. Construct two lines through this point that are tangent to the circle.
2. Draw a circle of radius of 4cm and select any point on the circle. Construct the line tangent to the circle at this point.

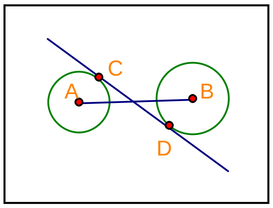
3. In the diagram NB and NA are tangents to circle M at points A and B.
(a) AB is a ______ of circle M.
(b) <MBN = ______ degrees. (Give your reason)
(c) <MAN = ______ degrees. (Give your reason)
(d) If MA = 6 cm, then MB = ____ . (Give your reason)
(e) What kind of triangle is [triangle] AMB. (Give your reason)
(f) What can you deduce about NA and NB. (Give your reason)
(g) If <BNA = 64°, calculate <BMA. (Give your reasons)
(h) If <BNA = 2x calculate <BMA. (Give your reasons)
(i) What type of quadrilateral is BNAM? (Give your reasons)



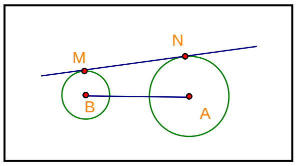
Discussion

1.A line that is tangent to each of two coplanar circles is called a **common tangent**.

(a) A common **internal tangent** intersects the segment joining the centers.



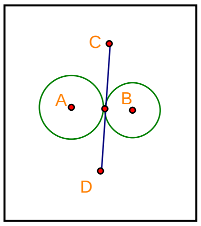
(b) A common **external tangent** does not intersect the segment joining the centers.

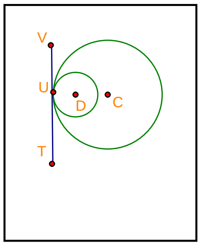


A circle can be tangent to a line, but it can also be tangent to another circle.
Tangent circles are coplanar circles that are tangent to the same line at the same point.

(a) [Circle]A and [circle]B are *externally* tangent.

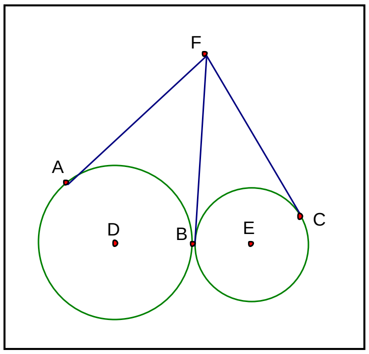
(b)[Circle]D and [circle]C are *internally* tangent.





3. **Example 1:**

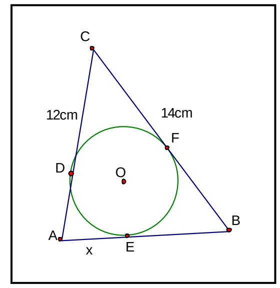
In the diagram below**,** FA, FB, and FC are tangents.
If FA = 12cm, what is the length of FC.



FA =FB …….. (tangents drawn from common point)
but FB = FC ….((tangents drawn from common point)
∴ FA = FC
but FA = 12 cm …. (given)
∴ FC = 12cm.

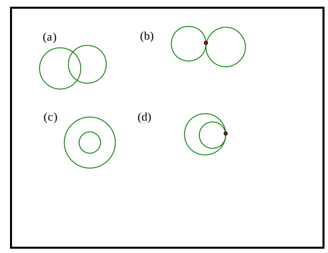
4. **Example 2:**

In the figure at the right, AC = 12cm, BC = 14cm and AE = x cm. Determine x , if each side of [triangle] ABC is a tangent to circle O.



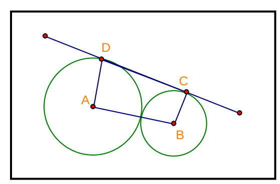
AD =AE = x (tangents drawn from common point)
Then , EB = 16-x and DC = 12 –x,
Also, FB = 16 –x and CF = 12-x
But CB = CF +FB
∴ 14 = (12-x) + (16-x)
14= 28 -2x
14 = -2x
thus, x = 7 i.e x =7cm

Exercise 8.6 **LO3** **AS 3.2 (f)**

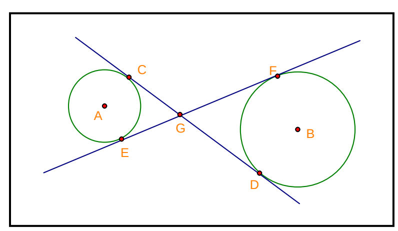


1.(a) How many common tangents could be drawn to each pair of circles?
(b) Which pair of circles shown alongside are externally tangent?
(c) Which pair of circles shown alongside are internally tangent?

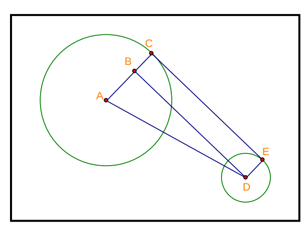
2. DC is a tangent to circle A and circle B. <ABC = 106°. Determine the size of <DAB.



3. CD is a tangent to the circles at C and D. EF is a tangent to the circles at E and F. If CG = 5cm and FG = 7 cm, determine the length of EF.

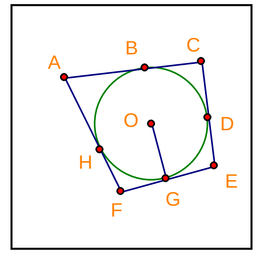


4 . In the figure above, CE is an external tangent to circles A and D and BD id perpendicular to AC.
(a) If DE = 6cm, AC =10cm and AD = 20cm,
(b) then determine the length of CE.
(c) If CE = 8cm, AD = 10 cm and DE =1cm,



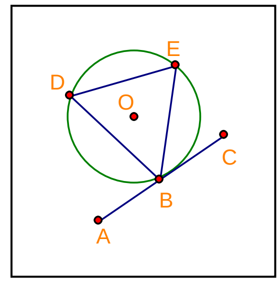
(d) determine the length of AB and
AR.

5.Quadrilateral AGEC is circumscribed about circle O. BC = 3cm, ED = 4cm , GF = 2,5cm and CF = 6cm. What is the perimeter of ACEG.



6. What is the greatest number of non-overlapping circles that can be drawn externally tangent to a given circle? All circles are of the same size.

FOR DISCUSSION



Point B is known as the point of ____________.
<EBC is known as an angle between the tangent ABC and the chord EB at the point of contact B.
<D is an angle subtended by chord EB in the alternate segment.
Note that <EBC and <D are on opposite sides of EB.

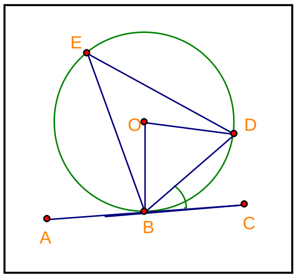
Now write down:
(a) the angle between the tangent ABC and the chord DB at the point of contact B, and
(b) an angle subtended by chord DB in the alternate segment.

Bigger Picture
In Chinese philosophy, all things are divided into two natural principles, yin and yang. Yin represents the earth, characterized by darkness, cold, or wetness. Yang represents the heavens, characterized by light, heat, or dryness. The two principles combine to produce the harmony of nature. The symbol for yin and yang is shown below. Construct your own yin and yang symbol. Start with one large circle. Then construct two circles with half the diameter that are internally tangent to the large circle and externally tangent to each other. Finally, construct small circles that are concentric to the two inside circles. Shade or colour your yin and yang symbol.

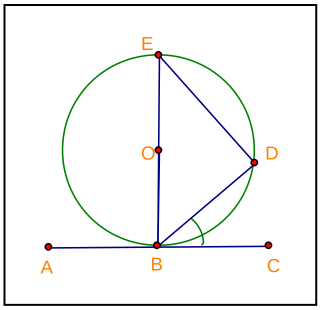
Picture of yin and yang symbol.

Exercise 8.7

1.ABC is a tangent to circle O and ∠DBC = 34°.
Calculate:
(a) ∠BOD
(b) ∠BED

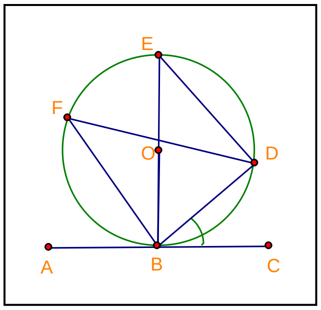


2.ABC is a tangent to circle O.
∠DBC = 50°.
Calculate ∠E.

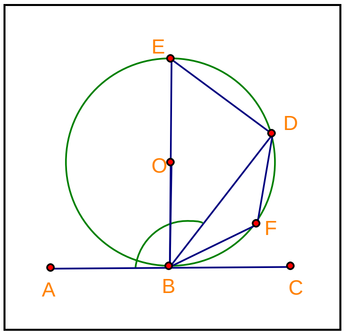


3. ABC is a tangent to circle O.
∠DBC = 46°.
Calculate ∠F.

.



4. ABC is a tangent to circle O.
∠ABD = 152°.
Calculate ∠F.



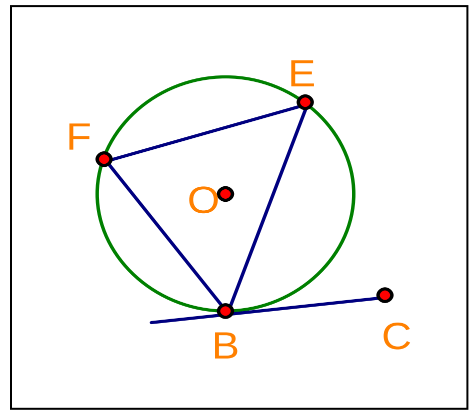
5. Generalise your findings in questions 1 to 4, i.e. express **in words** what you have discovered about the angle between a tangent to a circle and the chord drawn from the point of contact.

Compare your conjecture formulated in 5 with the conjectures of others in your class.

Discussion

1.You have probably formed the following conjecture in the preceding exercise.

Conjecture 12*:* The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment of the circle .(***Tangent-chord conjecture***).

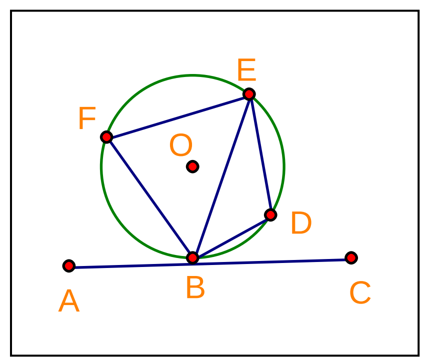


Given: Circle 0 with BC a tangent at B and
EB a chord subtending angle F.

Conclusion: <EBC = <F.

Reason: tan-chord conjecture or < in alternate segment

2. ABC is a tangent to circle O. <CDB = 37°
and <BDE = 112°. In the triangle BFE, BF = EF.

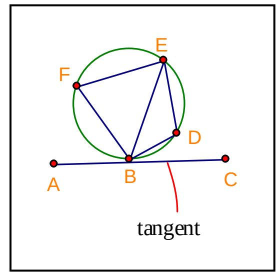


(a)	<ABE = <BDE	(tan-chord conjecture)
	but <BDE = 112°	(given)
	therefore <ABE = 112°	
(b)	<CDB = <BFE	(tan-chord conjecture)
	but <CDB = 37°	(given)
	therefore <BFE = 37°	
(c)	<BFE = < FEB	(BF =EF)
	but <BFE = 37°	(proved above)
	therefore < FEB= 37°	
(d)	<ABF = < FEB	(tan-chord conjecture)
	but < FEB= 37°	(proved above)
	therefore <ABF = 37°	

Note: The above example illustrates the use of deductive logic, which is also referred to as deductive reasoning.

Exercise 8.8

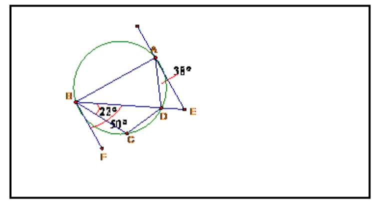
1. Apply the Tangent-chord conjecture to complete the following statements:



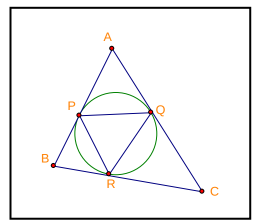
   (a) ∠ ABD = ∠ …
   (b) ∠ ABE = ∠ …
   (c) ∠ BEF = ∠ …
   (d) ∠ CBE = ∠ …
   (e) ∠ CBD = the angle subtended by arc …
   (f) any angle in minor segment BF = ∠ ABD = ∠ … between tangent …. and chord ….

2. AE and BF are tangents of the circle.
   ∠EAD = 38°, ∠CBD = 22° and ∠FBD= 50°.

   Find the angles of quad. ABCD



3. APB, AQC and BRC are tangents of the circle.
   ∠BCA = 48and ∠BCA = 50°.
   Find the angles of ΔPQR.



4. AB =5cm. See if you can invent a method for constructing a segment of a circle to contain angle ACB = 62°.
   Hint: Use the tan-chord conjecture.
5. (a) Write down the converse of the **Tangent-chord conjecture** in your own words.

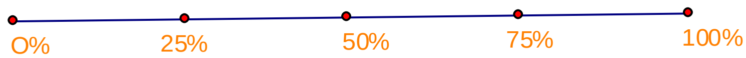
   b) If you believe your converse of the Tangent-chord conjecture is true, provide examples to support your view.

TRACKING MY PROGRESS

1. Explain the difference between each of the following pairs of terms. Use sketches to help you explain.
   (a) congruent circles and concentric circles.
   (b) Chord and diameter
   (c) Minor segment and major segment
   (d) Angle at centre and angle at circle.

(e) Tangent and secant.

2. Make a list of all the conjectures you have formulated in this chapter.
   How certain are you that each of your conjectures are always true?
   For each conjecture, record your level of certainty on the number line and explain your choice.

   You may draw a separate number line for each conjecture.

3. In this chapter you had the experience of using your conjectures to solve numerical riders. How do you feel about your ability to solve numerical riders using your established conjectures?
4. Select a rider in this chapter that you experienced difficulty with, but finally succeeded in completing it. Write down the difficulties you experienced and how you managed to overcome the stated difficulties.