Unit 9: Geometric Proofs and Rider Strategies

Introduction

In this chapter we re-visit the conjectures that we formulated in UNIT 8. We attempt to provide a logical argument to explain or establish the truth of (or verify) each established conjecture. A logical argument not only supplies us with understanding of why something is true, but can also help us establish the general validity of a result. In mathematics a logical argument whose purpose is explanation or verification is normally called a *proof.* In essence we will prove the established *conjectures* by providing a logical, reasoned argument using only definitions, axioms and proven theorems, which show that the stated conjecture or result does in fact hold true in all cases. In most instances, the conjectures are treated as routine riders in this book. Remember a rider is ……

Most conjectures and riders are proved by means of the method of direct proof. In a direct proof, we begin with what is given and by a logical chain of arguments deduce the required conclusion. Some of the rider strategies associated with direct proof are: direct application of theorems; congruency approach; algebraic approach.

Proof by contradiction is an example of indirect proof. In this chapter we use this strategy to prove the converses if some conjectures.

Although we have chosen to follow a particular approach in establishing our theorems, axioms and corollaries, it is permissible to follow any other logically correct order.

UNIT 9.1: Rider strategies

FOR DISCUSSION: Some general advice on doing geometric proofs:

1. Remember that in proving a rider you are not only proving it to your own satisfaction but so that your reader can follow your argument easily. Make your final proof clear.
2. Draw a large, clear, carefully labeled diagram. If your A’s look like D’s and your D’s look like O’s, everyone including yourself, will be confused.
3. Indicate with suitable markers on the sketch **all** the information that is given to you.
4. Make sure that you know what you have to prove and choose some strategy – i.e. **make a plan**.
5. Make a preliminary test to see whether your chosen strategy has some chance of success, and then follow it through.
6. Continue to indicate by suitable markers on the sketch what you have proved as you go along.
7. If you are stuck, it often helps to look **again** at the given information and to ask yourself: “ What use can I make of **that** fact ?’
8. If your strategy fails, **try again**.

[NB: Dear READER I have been writing angle A as <A , which is not the correct symbol , just for convenience. I will correct it later using equation editor as it is time consuming. So do not worry about that for now but rather focus on the mathematical procedures and processes]

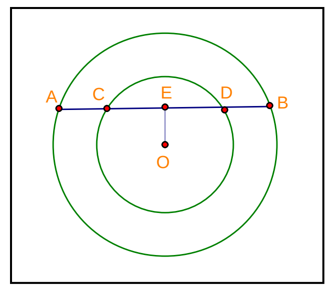
INVESTIGATION 9 A: Rider strategies

You may experienced the following three main strategies: direct application of theorem(s);the congruency approach and the algebraic approach**.** The examples below will help you remember these strategies. In addition to these rider strategies we will introduce the following rider strategies: use of other branch of mathematics; proof by contradiction, analysis.

A first rider strategy- direct application of theorem(s)

This is best illustrated by example.

Example:
In the accompanying figure, O is the centre of two concentric circles. Chord AB of the greater circle cuts the smaller circle at C and D. Prove that AC = DB



Proof
Draw OE $\perp$ AB
$\therefore$ AE = EB (perpendicular from centre to chord)
and CE = DE (perpendicular from centre to chord)
$\therefore$ AE –CE = EB- DE
$\therefore$ AC = BD

A second rider strategy - the congruency approach

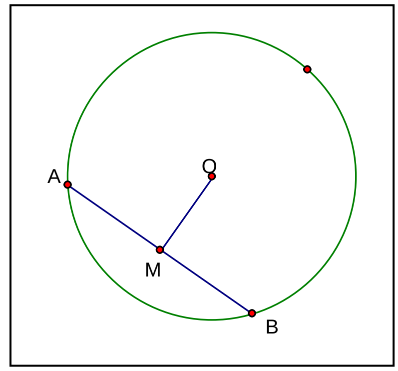
When do we use this approach? When we wish two prove that two sides or two angles are equal.
How do we know when we can use this approach?
If the two sides or two angles concerned are, or can be placed, in two separate triangles which might be proved congruent.

You have already had experience in this approach and have proved a theorems in this way. Below we refer to some of the conjectures you made in Chapter 8.

(a) You are now probably quite convinced, after investigation 8B, that the line drawn from the center of the circle, perpendicular to a chord always bisects the chord. Further exploration on the same (or other) sketch would probably succeed in convincing you more fully, but it really provides no explanation;

it merely keeps on confirming the statement's truth. Instead, we will now try to explain your conjecture as a consequence of other, more basic geometric ideas that you already know.

It is customary in geometry to provide reasons for each step in our explanation.



Here are some hints for planning a possible explanation based on the sketch on the right that you worked with.

1. Construct OA and OB.
2. What is the relationship between OA and OB? Why?
3. What is the **given** relationship between < OMA and <OMB?
4. Which side is common to both ΔOMA and ΔOMB?
5. What can you conclude from questions 2-4 regarding ΔOMA and ΔOMB?
6. Which case of congruency did you use to explain why ΔOMA and ΔOMB are congruent?
7. From question 5 or question 6, what can you conclude about MA and MB.
8. Summarize your explanation of the Perpendicular from the Centre to Chord Conjecture (i.e. Conjecture 1). You can use Questions 2-7 to help you. You might write your explanation as an argument in paragraph form or as a two column proof.

Your explanation in Question 8 explains Conjecture 1. It also establishes the general validity of your conjecture. that is, it shows that the result holds for all cases that satisfy the given conditions. Hence, w e say that you have written the proof of Conjecture1. Now that we know that our conjecture is valid for all cases, we will call it a **theorem** and refer to it as **Theorem 1** in this textbook.

Theorem 1: The line drawn from the center of the circle, perpendicular to a chord always bisects the chord. .(*Perp. from centre to chord)*

Note: You have used the congruency approach to prove Theorem1.

Exercise 9.1

1. Use the congruency approach to prove Conjecture 2, which was established in

Investigation 8B.

2. Use the congruency approach to prove Conjecture 3, which was established in Investigation 8B.

Once, you have written a proof for Conjecture 2 and Conjecture 3 you may then call it Theorem 2 and Theorem 3 respectively, in this textbook i.e.

Theorem 2: The line drawn from the center of the circle, perpendicular to a chord always bisects the chord.(*Mid-point chord, or line through centre and midpoint)*

Theorem 3: The perpendicular bisector of a chord passes through the centre of the circle.*(Perp. bisector of chord)*

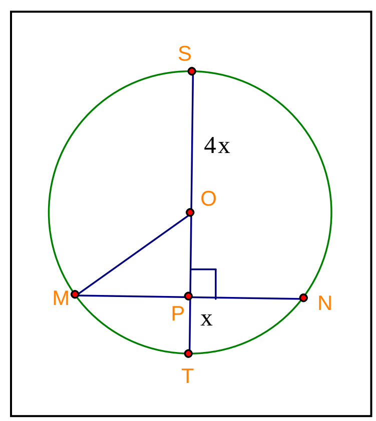
FOR DISCUSSION: More Rider strategies

Employed: When we wish to prove a relationship between the size of angles ot the length of lines, , e.g. <A = 2<B, <C = 90° - <D, < P = <Q; MN = 4 PT, etc.

Remember: Relationship between angles help us to prove other relationships, e.g. that two lines are parallel or that a triangle is isosceles, etc.

Below we apply this strategy to a geometric result.
Example 1

In the diagram alongside, MN is a chord of the circle with centre O. Diameter ST is perpendicular to MN at P. If PS = 4 PT, prove that MN = $\frac{8r}{5}$, where r is the radius of the given circle.



Strategy Chosen: Algebraic approach

Proof:

Let PT = x, then PS = 4x
∴ ST = 5x

SO = OT = 2½ x (radius = ½ diameter)
∴ OM = 2½ x

OP = 2½ x - x
= 1½ x

$MP^2 = MO^2 - OP^2$ (Pythagoras)
$= (2\frac{1}{2}x)^2 - (1\frac{1}{2}x)^2$
$= \underline{25x^2} - \underline{9x^2}$

$$= \frac{16x^2}{4}$$

$$= 4x^2$$

$\therefore$ MP = 2x
$\therefore$ MN = 2MP (perpendicular from centre to chord)
= 4x

$\therefore$ MN = 4 x r/2½ ($x = \frac{2\frac{1}{2}x}{2\frac{1}{2}} = \frac{r}{2\frac{1}{2}}$)

= 8/5 r

Some will argue that this example can be proved by direct methods – i.e. strategy 1, that is true .Try it!

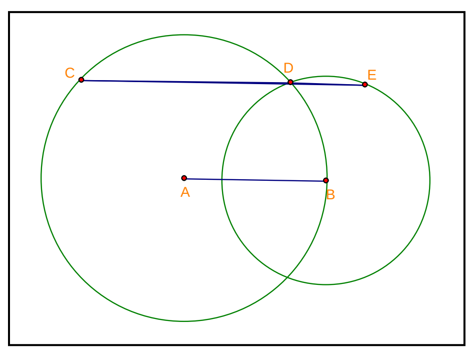
Advantages of the algebraic approach

1. Easy to express.
2. Easy to follow.
3. Every idea already proved is shown clearly on the sketch so that all interrelationships can be easily seen and can be used to prove the latter parts.
4. It is not necessary to see the whole plan through to the completion before starting; it is exploratory. (This is a major weakness in strategy 1)

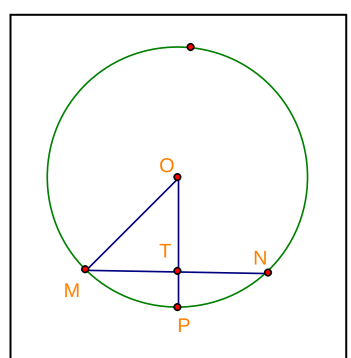
The first question often asked: " How do you know where to start? Where do you put the x?"
The answer: " It doesn't matter where you start, but obviously you should start where you have an advantage."

Exercise 9.2

1. In the diagram alongside, CE//AB. A and B are centers of the circles.
   Use the algebraic approach to prove that CE = 2AB.

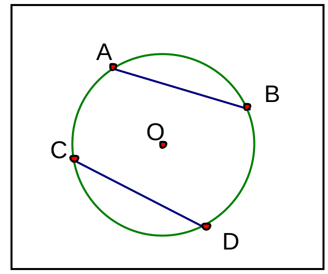


2. In ©O on the right, OM and OP are the radii. T is the Midpoint of chord MN.
   Calculate:

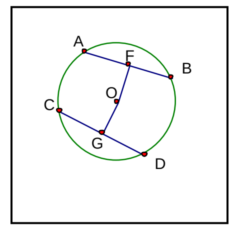


(a)OM, if OT = 5m and MN = 12 m.
(b)TP, if OM = 5m and MN = 8m
(c)OP, if TN = 6m and TP = 3m.

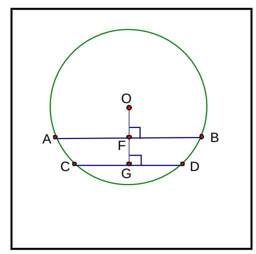
3. In the diagram on the right, AB and CD are equal chords of ©O. What can you guess about them? Prove it.



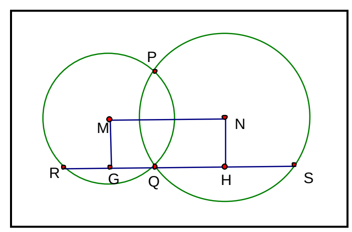
4.In the diagram on the right, AB and CD are equidistant from the centre 0. What can you guess about them? Prove it.



5. Circle O has radius of 50mm. Calculate the distances of chord AB = 80 mm and chord CD = 60mm from O.



6. In the figure, circle M cuts circle N at P and Q. PQS is a straight line drawn through Q such that R and S lie on the circles. MG is perpendicular to both MN and RQ and H is the midpoint of chord QC.
   (a) Prove that MB//RS
   (b) Prove that MN =1/2 RS



INVESTIGATION 9B: Using the algebraic approach to prove circle conjectures

After proceeding through investigation 8D(a), you probably formed the following conjecture :

Conjecture 4: *The measure of the angle subtended by the arc at the centre of the circle is double the size of the angle subtended by the same arc at the circle.*

Now let us try and explain your conjecture , using the ***algebraic approach***.

(a) In Diagram 1, in investigation 8D(a), you found the <AOC (at centre) is equal 2.<ABC (at circle) i.e Conjecture 4.

Now, use Figure (a) alongside and provide a logical explanation of Conjecture 4, using the algebraic approach:

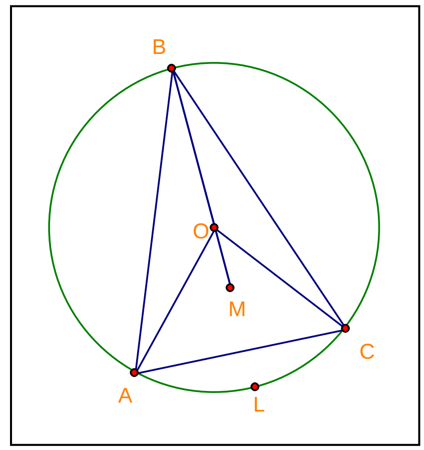


Figure (a)

Given: [circle] O with <AOC at the centre and <ABC at the circle on the same arc ALC.
To Prove: <AOC = 2.<ABC
Constr: Join BO and produce to M.
Proof: Let <ABO = x.
.: <BAO = x (AO = BO, radii)
.: <AOM = 2x (ext.< of Δ ABO)

Similarly let <OBC = y, ∴ <MOC = 2y

<AOC = 2x + 2y
= 2(x+y)
= 2.<ABC

b) In investigation 8D(a), you also found that the reflex <AOC (at centre) is equal 2.<ABC at circle. In other words, you found that Conjecture 4 still holds true. Now, use Figure (b) alongside and provide a logical explanation of Conjecture 4, using the algebraic approach. This will be a good indication of whether you understood the argument given in (a) above.

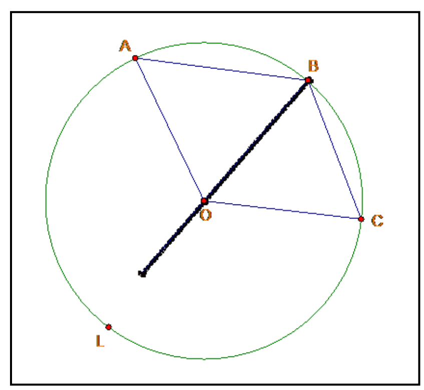
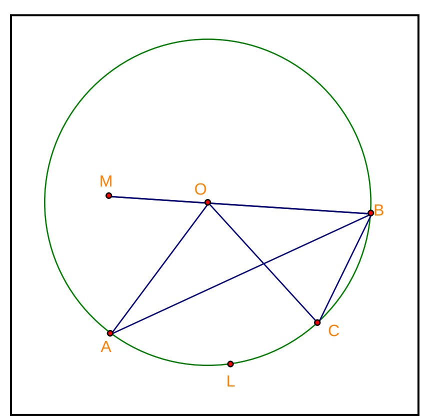


Figure (b)

c) In Diagram 3, in investigation 8D(a), you also found the relationship between <AOC (at centre) and <ABC (at circle) still hold i.e. you found that Conjecture 4 still holds true. Now, use Figure (c) alongside and provide a logical explanation of



Conjecture 4, using the **algebraic approach**.

Figure (c)

In questions a, b, and c you have provided a logical explanation of why your Conjecture 4 is always true i.e. you have provided a convincing proof to explain why Conjecture 4 is always true. Consequently, we will claim Conjecture 4 to be a **Theorem.** We will specifically, refer to it as the **angle at centre theorem,** in this book.

> **Theorem 4:** The measure of the angle subtended by the arc at the centre of the circle is double the size of the angle subtended by the same arc at the circle. *(< at centre = 2 x < at circumference)*

A **corollary** is an important logical deduction or result made from an axiom or theorem with little or no proof. In Exercise 8.5 , you probably made the following deductions in questions 5,6 and 7 respectively

(i) Equal arcs of a circle subtend equal angles at points on the same circle.
(ii) Equal arcs or chords of a circle subtends equal angles at the centre of the circle.
(iii) Chords or arcs of a circle are equal if they subtend equal angles at the circle.

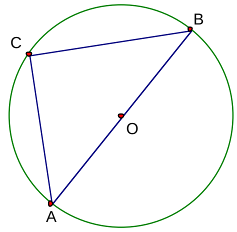
The above deductions are actually the corollaries to Theorem 4.
You should try to prove each of the corollaries in detail.

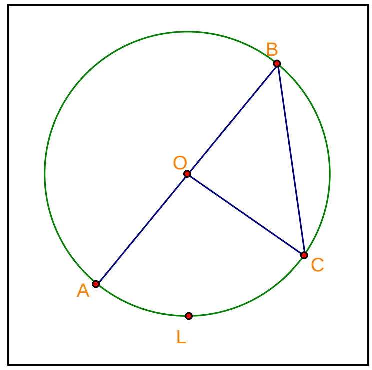
Furthermore, in No1 of Exercise 8.5, you realized that <ABC = 90° and you have utilized Theorem 4 to explain why your conjecture is always valid. By doing this, you have actually proved the following corollary:

Corollary 4: The angle at the circumference of a circle subtended by a diameter is a right angle
or
The angle in a semi-circle is a right angle.*(< in semi-circle)*

State the converse of Corollary 4 in your own words.

Note: The converse of Corollary 4, may be utilize to identify a diameter of a circle or prove that a given line segment is a diameter of a given circle, i.e if <ACB = 90° in the diagram, then AB is a diameter of the circle.





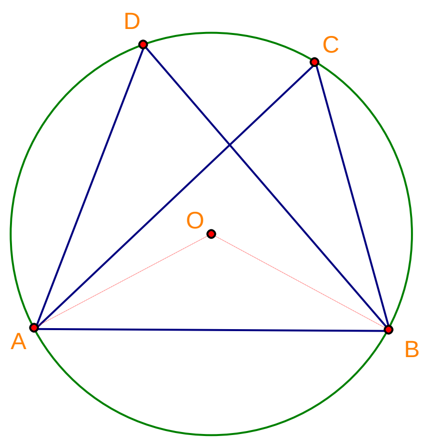
Try this…

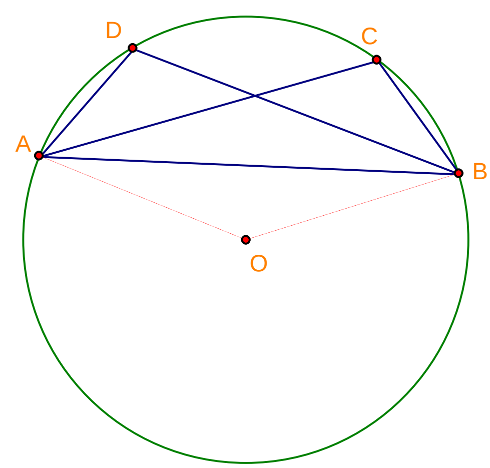
Does the relationship between <AOC (at centre) and <ABC (at circle) still hold in this sketch alongside?
Explore the relationship and prove your finding, using the algebraic approach.

INVESTIGATION 9C: Using the direct application approach to prove a circle Conjecture

In investigation 8D(b), you probably formed the following conjecture:
Conjecture 5: *The angles subtended by a chord at the circle on the same side of the chord are equal.*

We can prove that his conjecture will always be the case, through the **direct application** of Theorem 3. We have started with proof below. Copy the outline and complete the proof.





Given: Angles ACB and ADB in the same segment ADCB of a circle centre O.
To prove: <ACB = < ADB
Construction: Draw OA and OB
Proof: <AOB = 2.<DC Why?)
and <AOB = ……. (Why?)
Therefore 2.<C = ……….. (Why?
Therefore <C = ……

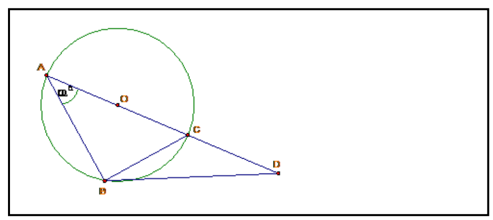
Now that we have proved that Conjecture 5 is always the case, we will regard it as a theorem in this textbook i.e.

Theorem 5: The angles subtended by a chord at the circle on the same side of the chord are equal.(*<'s on same side of chord)*

Theorem 5 is usually referred to as the **angles in the same segment theorem**, which is commonly abbreviated as ***(<'s in same segment)***

INVESTIGATION 9D*:* A fourth rider strategy – use other branch of Mathematics

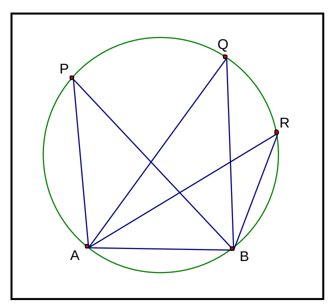
Example:
In the figure alongside, AOC is a diameter of the circle.
ACD is a straight line and CD = CB = x , ∠BAC = m.
Prove that $BD^2 = 2x^2(1 + \sin m)$



Proof:
<ABC = 90°…. (angle in semi ©)
<ACB = 90°-m….(interior angles of ΔA BC)
< BCD = 90°+ m… (straight line ACD)

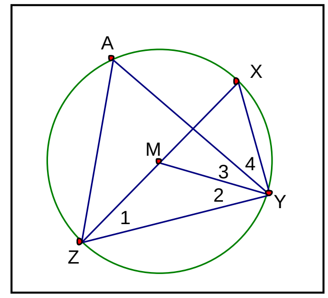
$$BD^2 = CD^2 + BC^2 - 2\,CD.BC.\cos \angle BCD \qquad (\text{In } \Delta\ BCD)$$
$$= x^2 + x^2 - 2\,x.x.\cos(90° + m)$$
$$= 2x^2 - 2x^2(-\sin m)$$
$$= 2x^2 + 2x^2 \sin m$$
$$= 2x^2(1 + \sin m)$$

Exercise 9.3

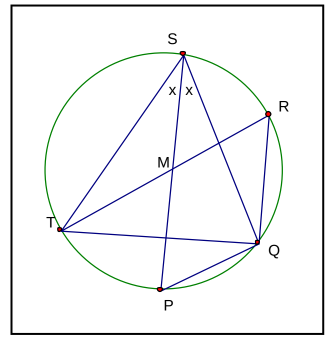


1..In the accompanying figure, P, Q, R, B,A are point on a circle.
Give reasons why
(a) <P = <Q = <R
(b) <PBQ = <PAQ
(c) <QAR = <QBR

3. In the accompanying figure, XY is a diameter of the circle WXYZ Whose centre is at the point M. I <WZx = 26° and <ZMY = 98°, Calculate giving reasons, the size of each of the following angles:
   (a) <W
   (b) <Z_1
   (c) <Y_3



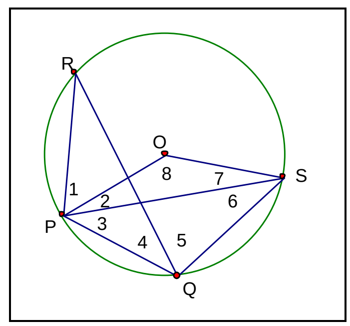
4.In the accompanying figure P,Q, R, S and T are point of a circle. PS bisects <TSQ and QR//PS. PS and TR intersect at M.
Prove that: (a) PQ//TS and
(b) <SMR = <TSQ



5. In the figure alongside, o is the centre of the circle and P, Q, R and S are points on the circumference such that PQ = QS.

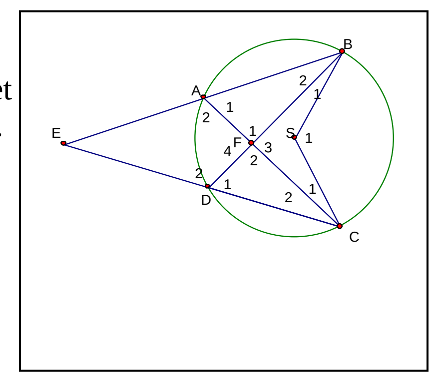
   If < O_8 = x, express the following angles in terms of x. Give reasons for your answers.

(a) < P_2
(b) <PQS
(c) <R



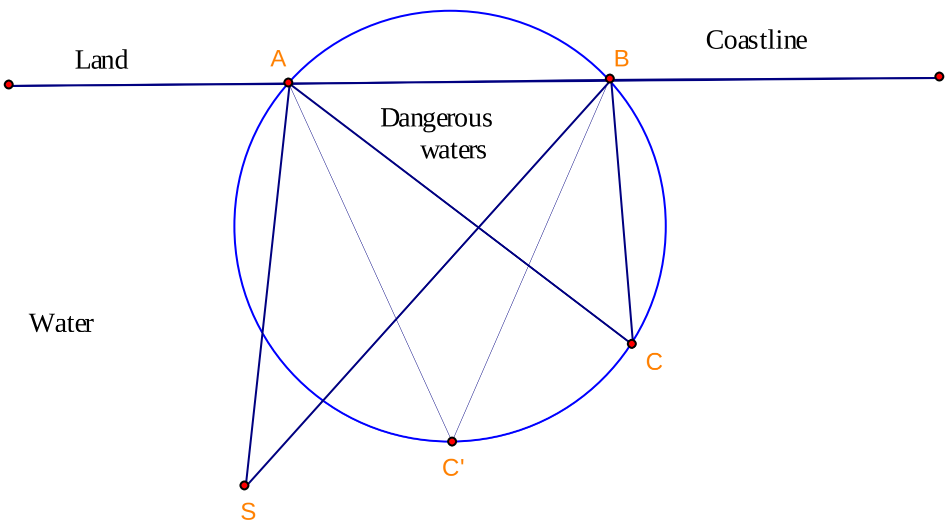
6. In the diagram on the right, A, B and C are on the circle with centre S. Chords BA and CD are produced to meet at E. AC and BD intersect at F and SB are and SC are drawn. <ABD = x and <BAC =y.

   a. Express <DFA in terms of x and y.
   b. Prove that: <BSC = <DFA + <E



Bigger Picture

As a ship nears the coastline, it must avoid dangerous rocks and shoals. Based on sailor's observations, the hazardous waters are charted. The region is represented by a circle, as shown below. The circle passes though lighthouses A and B. Point C is any other point on the part of the circle passing through the water.



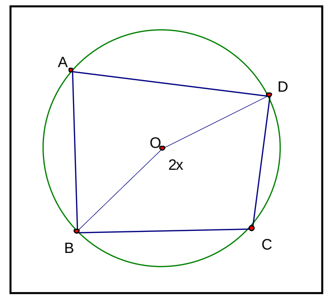
1. < C is called *the horizontal danger angle*. In geometry, we say that <C is a)n) _____ angle.
2. Use your protractor to find the measure of <C.
3. Why is the measure of <C the same for all locations of C? For example, why does <C' = <C.
4. Let S be the location of the ship. Instruments mounted on the ship can determine the measure of <ASB. Measure <ASB. How does it compare with the measure of <C.
5. Trace the diagram above. Choose another point S' outside the circle. Measure <AS'B. How does its measure compare with the measure of <C.
6. Now choose two points S1 and S2 inside the circle to represent the ship sailing in dangerous waters. Measure <AS1B and <AS2B. How do these measures compare with the measure of <C?
7. Explain how sailors use the horizontal danger angle to keep out of dangerous waters. (Hint: Consider your answers to questions 4,5 and 6).

UNIT 9.2: Cyclic quadrilaterals

Investigation 9E: Cyclic quadrilateral properties

Describe what a cyclic quadrilateral is, over the phone to someone who is not yet acquainted with it. Share this telephonic description with your peers in your own class. Then ask your peers to construct a figure that complies with your description to see if it really gives the desired figure. Compare the results of yours peers. DO you need to change your original description?

Copy and complete the following Conjecture you established in chapter 8:
The opposite angles of a cyclic quadrilateral are ___________.



We will now use the **algebraic approach**, to explain and establish the validity of the above conjecture. We will use the diagram on the right.

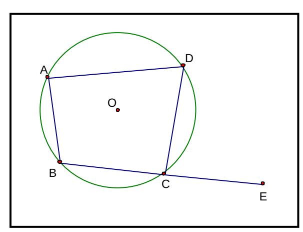
1. Write down the given information.
2. Write down what you are required to prove.
3. Describe the constructions needed in order to facilitate an explanation of the required proof.
4. Let's start by letting <BOD = 2x.
5. Express reflex <BOD in terms of x. Give a reason for your statement.
6. Express reflex <A in terms of x. Give a reason for your statement.
7. What is the relationship between reflex <BOD and <C . Why?
8. Now express <C in terms of x.
9. What can you conclude about the sum of <A and <C.
10. What can you conclude about the sum of <ABC and <ADC? Why

Look over questions 1-10. Now write a two-column proof of your conjecture in your own words, which characterizes the algebraic approach. You might include a demonstration sketch to support and explain your proof. Once you have completed the proof, then only may you refer to the given conjecture as a theorem.
Note, you could prove this theorem by using direct methods – i.e strategy 1.Try it!.

Theorem 7:
The opposite angles of a cyclic quadrilateral are supplementary.*(Opposite <s of cyclic quad.)*

A **Corollary**, i.e., direct consequence of Theorem 7, is that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

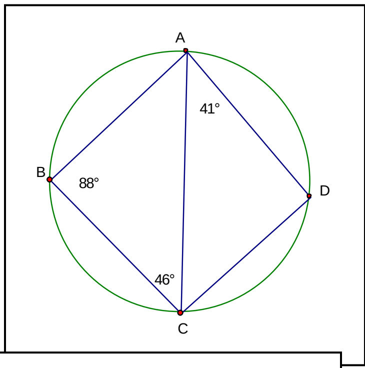
Corollary: The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



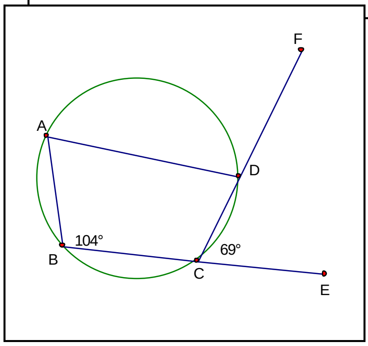
i.e. <DCE = <A

Note: that this corollary is actually our conjecture 10, that we formed in Investigation 8E(d).

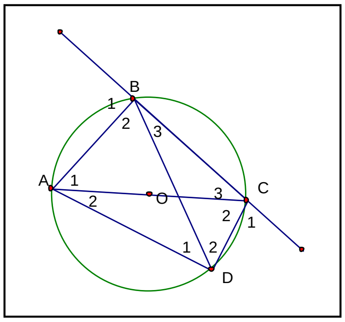
Exercise 9.4



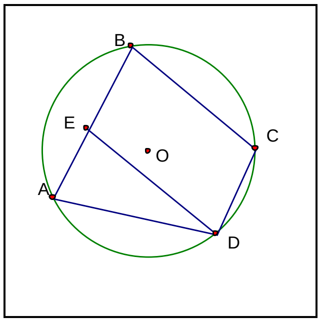
1. In the accompanying figure, ABCD is a cyclic quadrilateral. < B = 88°, < ACB = 46° and <CDA = 41°. Determine with reasons, the size of:
   (c) <ADC
   (d) <ACD



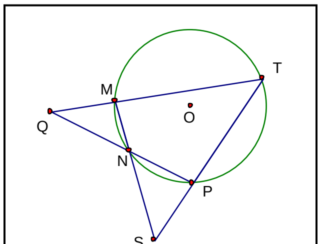
2. In the accompanying figure, DCE is an exterior angle of cyclic quadrilateral ABCD.<ABC = 104° and <DCE = 69°. Determine with reasons the size of :
   (e) <BAD
   (f) <ADF



3. In circle O, AOC is a diameter and the four points A,B,C and D lie on the circle . < D_1 = 37° and
   (a) Explain why AB is perpendicular to BC.
   (b) What kind of quadrilateral is ABCD? Give a reason for your answer.
   (c) Calculate:
      (i) < C_3
      (ii) < A_1
      (iii) < B_3
      (iv) < C_1
4. In the figure on the right, cyclic quadrilateral ABCD has AB//CD. DE//BC meets AB at E and < B = 74°.
   (d) Calculate < DAE.
   (e) Prove DE = AB



4. MNPT is a cyclic quadrilateral. TM and PN produced meet at Q while



MN and TP produced meet at S. If <TMN = x°, TPN = y° and
<P + <Q = 90°, prove that x + y = 270° – 2<T.

UNIT 9.3: Proof by contradiction

For Discussion: Indirect reasoning

A lawyer, summing a legal case, says, "The prosecutors assume the defendant – my client – committed the crime. Then my client would have been at the scene of the crime. But remember we brought in witnesses, minutes of a meeting and telephone records. These extra people and records demonstrate my client was on a farm 300 kilometers away. A person can't be in two places at one time! My client could not have been both on the farm and at the scene of the crime at the same time. So the assumption that my client committed the crime cannot be true. So it must be false. The honorable judge the defendant is not guilty."

In summing up, the lawyer has used indirect reasoning.

In **direct reasoning**, a person begins with given information known to be true. The necessary laws, algorithms, axioms, definitions, theorems and corollaries are used to reason from that information to a conclusion. The proofs that you have written so far in this book have been direct proofs. In each case you have started with given information that is true.

In **indirect reasoning**, a person tries to rule out all the possibilities except the one thought to be true. The lawyer quoted above has used this approach to argue that the defendant could not have been at the crime scene. This is exactly what you do when you solve logic examples like in No1. of the exercise below.

You can rule out the possibility if you know it is false. But how can you tell that a statement is false? There are two ways to do this:

1. One way to tell is if you know its negation is true. For example, suppose you know y = 5 is true. Then y not equal to 5 is false. Suppose you know that ∆ABC is isosceles. Then ∆ABC is scalene is a false statement.

2. You also know a statement is false if it contradicts another statement known to be true. For instance, if your fiend Lillian is a senior, then she cannot be your junior. If you know 5 is a solution to an equation, and there is only one solution to that equation, then 3 cannot be a solution. If a statement contradicts another, they are called **contradictory** statements.
   In other words, two statements are contradictory if and only if they cannot both be true at the same time

Example 1: Let p be the statement: <ABC is acute.

Let q be the statement: <ABC is a right angle.
Are p and q contradictory statements?

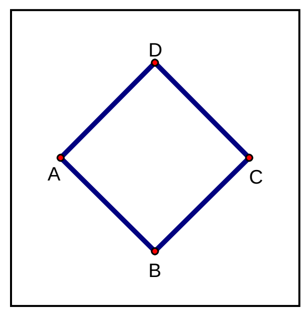
Solution: Yes, p and q contradictory statements.
An acute angle has measure less than 90°. A right angle has measure 90°.
A number cannot be both less than 90 and 90 at the same time. So an angle cannot be both acute and right at the same time.

Exercise 9.5

1. Carol, Sue, Dave, and Jim each play a different instrument in the school band. The instruments they play are clarinet, cornet, flute, trombone, and tuba. From the clues below determine which instrument each student plays.
   (a) Carol plays either the clarinet, cornet, or tuba.
   (b) Sue does not play the flute.
   (c) Dave does not play any of these instruments: trombone, cornet, flute, or clarinet.
   (d) Jim plays either the tuba or the cornet.

2. Give an example of two contradictory statements, and explain why they are contradictory.

3. Let p : ABCD is a rhombus. Let q: ABCD is a rectangle. Are p and q contradictory statements? Give an explanation for your response.



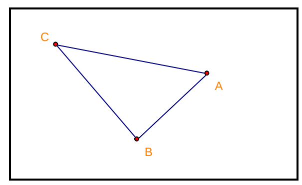
Bigger Picture:

Sometimes it isn't so easy to tell whether a statement is true or false. Then you can employ the logic used by the lawyer in the situation described at the beginning of this lesson.

Step 1: If you think statement is false, start by assuming it **for the moment** and reason from it. (The prosecutors thought the defendant was at the spot the crime was committed. The lawyer started with this assumption.)
Step 2: Using valid logic, try to make the reasoning lead to a contradiction or false statement. (The lawyer argued that the defendant would have had to be in two places at the same time.)
Step 3: Since the reasoning leads to a contradiction or other false statement, the assumed statement must be false. (The lawyer concluded that the defendant could not be at scene of the crime.)

This logic exemplifies the *Law of Indirect Reasoning*, which reads a follows:
If a valid reasoning from a statement *p* leads to a false conclusion, then *p* is false.

Example 1: Use an indirect proof to show that no triangle has two obtuse angles.



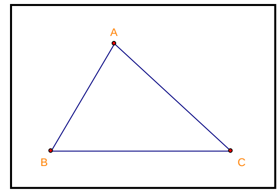
Step1: Assume both<A and <B of are obtuse.
Step2: Then, by definition of obtuse, $<A > 90^\circ$ and $<B > 90^\circ$. By the Addition Property of Inequality, $<A + <B > 180^\circ$. Then because $<C > 0^\circ$ for any angle in a triangle,
$<A + <B + <C > 180^\circ$. But the Triangle- Sum Theorem says that
$<A + <B + <C = 180^\circ$.
Step 3: The last two statements in step 2 are contradictory. A false conclusion has been reached.
By the Law of Indirect Reasoning, the assumption in step 1 is false. <A and <B of ΔABC cannot both be obtuse.

The above example exemplifies the use of **indirect reasoning which is the characteristic feature of an indirect proof**. This is a very powerful technique, which is used frequently in Mathematics to prove converses, is also commonly known as **proof by contradiction .** Others refer to the **indirect proof** technique as the **reductio ad absurdum strategy.** Its name speak for itself ("reduction to the absurd").

Exercise 9.6

1. State the Law of Indirect Reasoning.
2. When are two statements contradictory?
3. By indirect reasoning, show that a quadrilateral cannot have all four angles obtuse.
4. Use an indirect proof to show that no triangle can have two right angles.

5. Refer to ΔABC below.
   Given: <A> <B> <C.
   Prove: ΔABC is scalene.

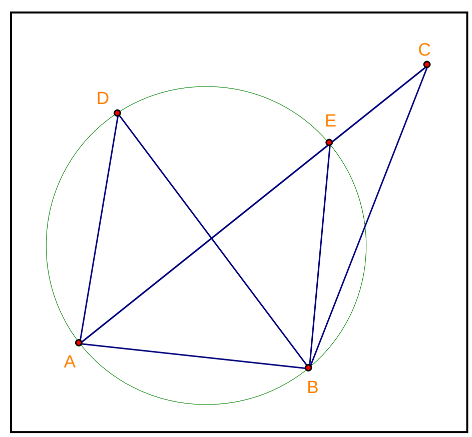


For Discussion: A fourth rider strategy
Now that you know about the method of indirect proof, we can investigate the use of proof by contradiction as a rider strategy.

A fourth rider strategy – Proof by Contradiction
In Chapter 8, the Conjecture 6, which is the converse of Conjecture 5, reads as follows: *If a line segment joining two points subtends equal angles at two other points on the same side of that line segment, then the four points are concyclic* (i.e they lie on a circle).

You can use a logical argument to verify this conjecture. A logical argument not only supplies us with understanding of why something is true, but can also help us establish the general validity of a result. A logical argument in mathematics whose purpose is verification is normally called a *proof*.

One method of proof is called *proof by contradiction.* In this kind of proof, you start by assuming your conclusion is false. Then you show that this leads to a contradiction, indicating that your conclusion must have been true. Below is an example of proof by contradiction for Conjecture 6.



Given: Line segment AB subtending equal angles ACB and ADB at two points C and D on the same side of line segment AB.
To Prove: Points A, B, C and D are concyclic (i.e. they lie on a circle)
Proof:
Step1: Suppose the circle passing through the points A,B,D does not pass through C but intersects AC or AC produced at E.
Step2: <ADB = <AEB (why?)
But <ADB = <ACB (why?)
Therefore <AEB = <ACB (why?)
But this not true since :
<AEB = <ACB + < EBC (ext.< of ΔEBC)
(Note: it is impossible for the size of an exterior angle of a triangle to equal to the size of one interior opposite angle)
Step3: The reasoning leads to a false statement.
Hence our assumption that the circle does not pass through C must be false.
Therefore A, B, C and D lie on the circle i.e. A, B,C and D are concyclic.

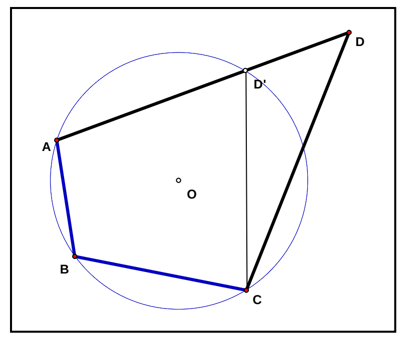
Now that you have established the general validity of Conjecture 6, we will consider it to be a theorem in this book.

Theorem 6: *If a line segment joining two points subtends equal angles at two other points on the same side of that line segment, then the four points are concyclic* (i.e they lie on a circle). *(Line segment subtends equal <'s on the same side).*

INVESTIGATION 9F: Proof by contradiction

Let us now consider Conjecture 9:
If the opposite angles of a quadrilateral are supplementary, that quadrilateral is cyclic.

We will use the accompanying figure to develop a **proof** for the above-mentioned conjecture using the proof by contradiction strategy.



So we start by assuming that the opposite angles of quadrilateral ABCD are supplementary, but quadrilateral ABCD is *not* cyclic.

1. What is the relationship between <ABC and AD'C? Why?
2. What is the given relationship between<ABC and <ADC? (this relationship does not match the measures in your sketch!)
3. What can you therefore conclude about <ADC and <AD'C?
4. Consider the exterior angle AD'C of Δ DCD'. Write an expression relating it to interior angles ADC and DCD'.
5. Compare your answers in Questions 3 and 4. What can you conclude from this ? What does your conclusion imply about D and D'.

Note: We assumed that what we had to prove was false and then showed that this led to a contradiction. So our assumption was false and what we had to prove was true.

Present Your Proof

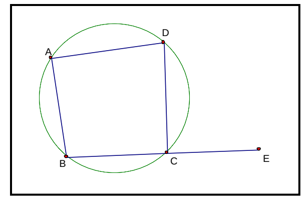
Look over questions 1-5. Now write a proof of your conjecture in your own words. You might include a demonstration sketch to support and explain your proof.

Note: Now using logic which exemplifies indirect reasoning, you have provided a convincing proof to explain why Conjecture 9 is always true. Consequently, we will claim Conjecture 9 to be a t**heorem.** We will specifically, refer to it as Theorem 8, in this book. Theorem 8 is actually the converse of theorem 7.

Theorem 8*:* If the opposite angles of a quadrilateral are supplementary, that quadrilateral is cyclic**.** *(Opp. <'s supplementary)*

The following **corollary** is a direct consequence of Theorem 8.

Corollary B : If an exterior angle of a cyclic quadrilateral is equal to its interior opposite angle, then the quadrilateral is cyclic.**(ext. < = int.op.<)**



We can summarize the results in Theorems 6,7 and 8 as ways of proving that a quadrilateral is cyclic. This summary will be usefull when proving riders in

Exercise 9.7.

TO prove that a quadrilateral is cyclic, prove that :

One pair of opposite angles is supplementary **or**

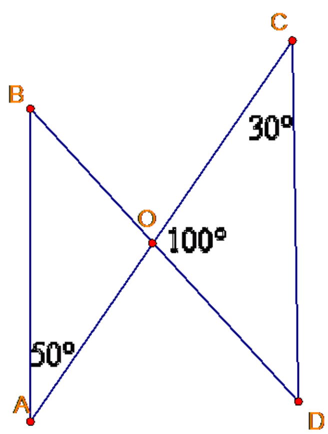
A line segment subtends equal angles at two points on the same side of it or

An exterior angle is equal to its interior opposite angle (optional)

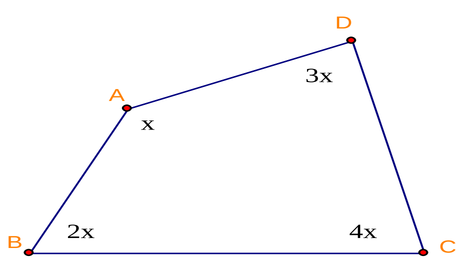
Exercise 9.7

Prove that A,B,C and D are concyclic in figures 1 to 3.

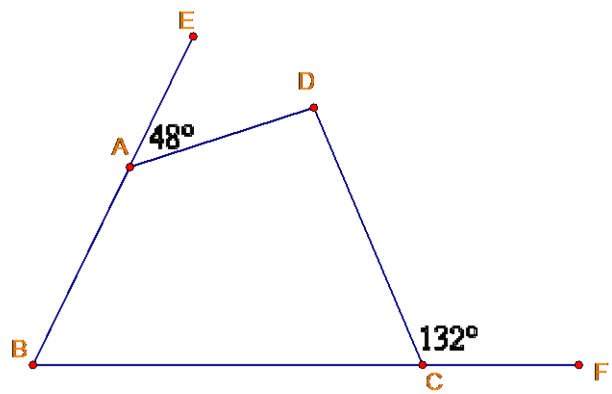
1.



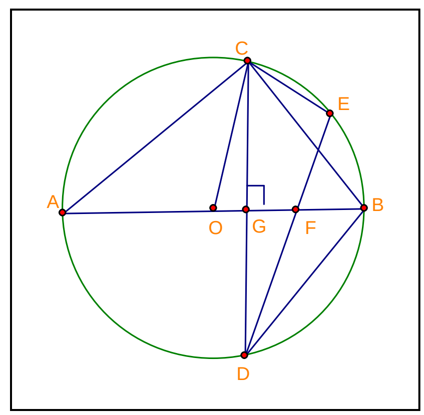
2.



3.



4. In the figure on the right, AB is a diameter of circle O. Chord CD is perpendicular to AB and cuts AB in G. Chord DE cuts AB in F. CE,CB,DB, OC and AC are drawn.
Prove that CEFO is a cyclic quadrilateral.



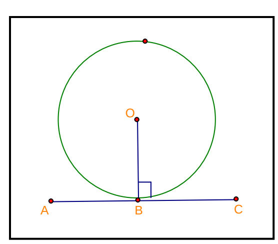
UNIT 9.3 : Tangents and secants

In Investigation 8G, you probably formulated the **tangent -radius** conjecture.
We will agree that there can be no argument or disagreement about this fact i.e. we agree that our conjecture is obviously true. Hence we, will consider this conjecture as an **axiom** in this book.

Axiom: A tangent to a circle is perpendicular to a radius at the point of tangency .
(Tangent $\perp$ radius)

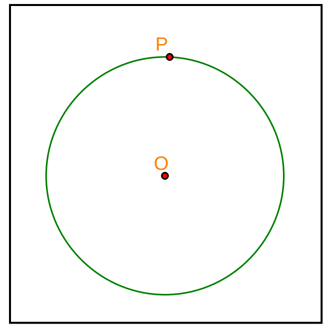
Exercise 9.8

1.In the figure, OB is perpendicular to AC in ©O. Prove that ABC is a tangent at B.

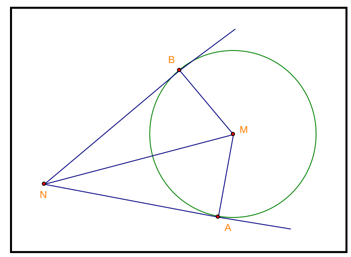


Note: You have just proved the that converse of the tangent-radius conjecture is true.

2.Using what you have proved in question 1, describe how you would construct a tangent to circle O at P.



3. In the accompanying figure, NB and NA are tangents to circle M at points A and B. Copy the figure, into your workbook, and use the congruency approach to prove that NB = NA



You have now proved Conjecture 12 is always true. Conjecture 12 will now will be referred to as a **theorem** in this book .

Theorem 9: Two tangents drawn to a circle from the same point outside the circle are equal**.(Tangents from same point)**

For Discussion: Using previous theorems to prove new results

1. You have probably formed Conjecture 13 as follows:

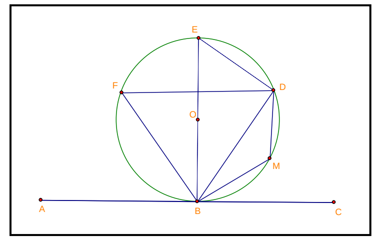
   The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment of the circle (**Tangent-chord conjecture**).

   To convince yourself that this conjecture does hold, you have used inductive reasoning – this means that you have drawn conclusions from your observations .

   However, if we use deductive reasoning, in other words, we use existing results in mathematics and logical argument, we can show that this result applies to all such cases, and not just the cases you have been engaged with. A logical proof will also help us to understand why the result applies to all cases. It may also convince you that the results do actually hold in all cases.

To continue with a proof of the conjecture, you can use a form of proof called *direct application of theorems.*

In the accompanying figure, ABC is a tangent to the circle at B. We will use the given figure to prove Conjecture 13.



Copy the outline of the proof. We give you the hints for some of the steps of the proof; you must fill in the gaps.

Given: Any [circle] O with chord BD and tangent ABC.
To Prove: (a) <DBC = <F
and (b) <ABD = <M
Construction: Draw the diameter BOE and join DE

Proof: (a) <EDB = 90° (why?)
Therefore <E + <B1 = ….. (why?)
But < DBC + <B1 = 90° (why?)
Therefore < DBC = < E (why?)
= <F (why?)

(b) <ABD + <DBC = 180° (why?)
<F + <M = 180° (why?)
therefore <ABD + <DBC = <F + <M (why?)
therefore <ABD = <M (why?)

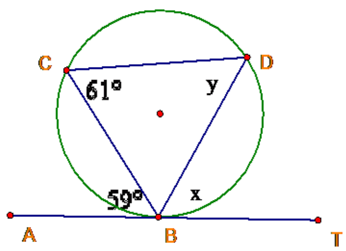
Through the appropriate application deductive reasoning using axioms and previously proven theorems, we have shown that our conjecture holds true in all cases. Can you identify any earlier results in the proof above?This conjecture (result) will be considered as a theorem, which is commonly known as the tan-chord theorem.

Theorem 10:
The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment of the circle. *(Tan-chord theorem)*

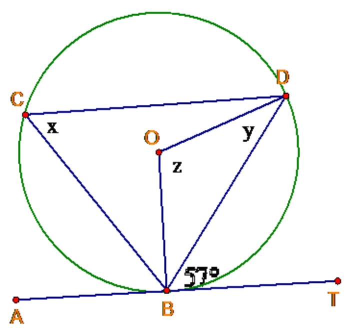
Exercise 9.9

1. Carefully study the following figures. Find the value o of the letters. In each case .O is the centre of the circle and BT is a tangent.

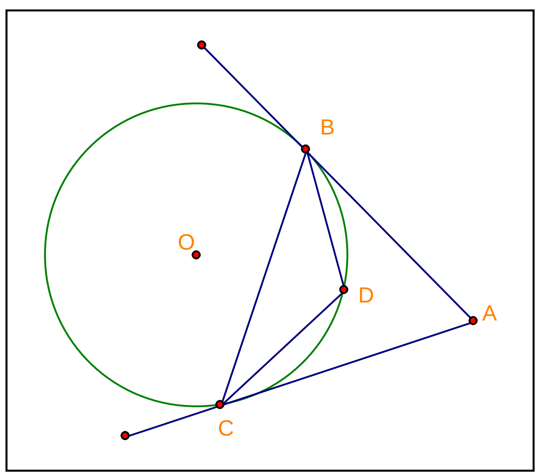
(a)



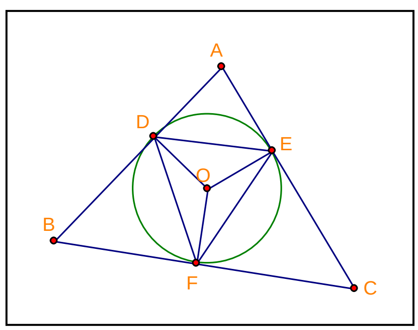
(b)



2.In the figure on the right, AB and AC are tangents to circle O. <DCA = 37° and <BCD =28°.
Calculate: (a) <BDC
(b) <BAC



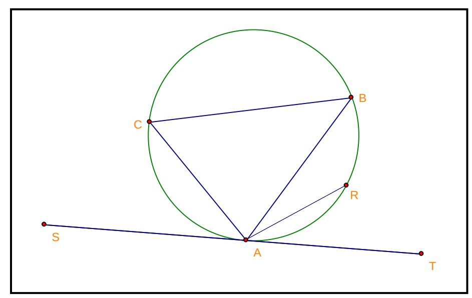
5. The inscribed circle of Δ ABC touches the sides of Δ ABC at A, B and C. If the size of <B = 64° and <C = 61°, find the sizes of the angles of Δ ABC



For Discussion : Using proof by contradiction to prove new results
In chapter 8, you wrote down the converse of the tangent-chord conjecture and provided examples to support your view. You probably formulated the converse of the tangent-chord conjecture as follows:
Conjecture 14: If a line drawn through the end point of a chord making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.

You can use a logical argument to verify Conjecture 14. A logical argument not only supplies us with understanding of why something is true, but can also help us establish the general validity of a result. A logical argument in mathematics whose purpose is verification is normally called a *proof*.

Through the use of the strategy called *proof by contradiction, we will demonstrate the general validity of the result announced in Conjecture 14.*
Before we continue with the proof by contradiction, write down the **three** sequential steps one needs to when you use this strategy .



Let us consider the accompanying figure , which we will use to explain why our conjecture is true in all cases.
In this given figure, AB is a chord of a circle and SAT is a line so that <BAT = <ACB in the alternate segment.

Copy the outline of the proof. We give you the hints for some of the steps of the proof; you must fill in the gaps.

Given: AB is a chord of a circle and SAT is a line so that <BAT = <ACB in the alternate segment.
To Prove: SAT is a tangent to the circle at A.
Proof: Assume that A is not a tangent; draw tangent AR
Now, <BAT = ……. (why?)
But <BAR = ……… (why?)
Therefore ….. = ……. (why?)
But this is impossible unless AR falls along ……
[Therefore] our assumption is false.
[Therefore] SAT is a tangent.

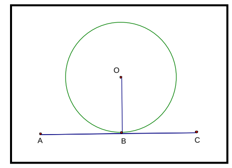
Now that we convinced ourselves about the validity of Conjecture 14, we will
1 consider it as a theorem in this book.

Theorem 11 :
If a line drawn through the end point of a chord making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.*(< between line and chord = < in alt.segment)*

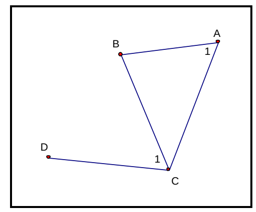
We can summarise our earlier results as ways to prove that a line is a tangent to a circle. This summary is useful for proving riders in Exercise 9.10.

Ways of proving that a line to a circle is a tangent

1.If ABC $\perp$OB, the radius of the circle, then ABC is a tangent to circle O.

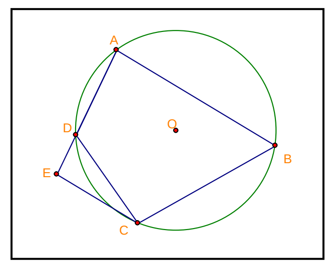


2.If <DCB = < BAC, then CD is a tangent to the circle passing through A,B and C.



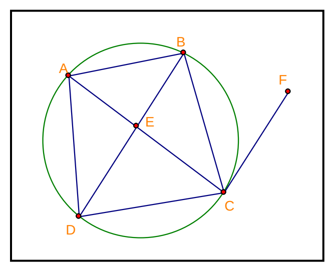
Exercise 9.10

1.In ©O on the right, and AB s parallel to EC. Prove that BC is a tangent to circle DEC.

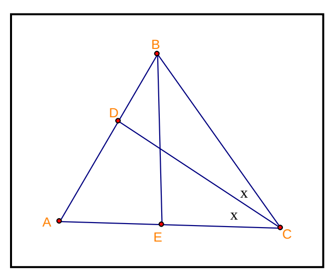


2.In the figure on the right, CF is a tangent and DB is Parallel to CF. Prove that

(a) CA bisects <BAD.
(b) CB is a tangent to circle BEA



3.In the figure on the right, DC bisects, BCA and BF = BD. Prove that CB is a tangent to circle BAE.



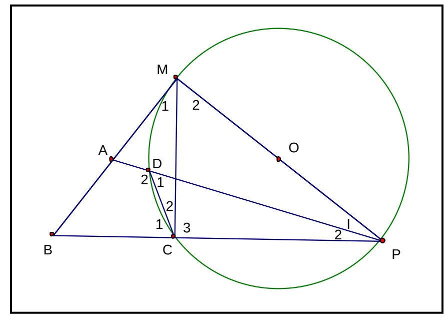
Discussion: A sixth rider strategy – analysis

In synthetic proof we start from what we know and then use theorems and logical arguments until we arrive at what we have to prove.

In analytic proof we work backwards from what we have to prove. Let us illustrate through an example

Example

In the accompanying diagram, MOP is a diameter of the circle with centre O . MAB is a tangent to the circle at M.AP and BP intersect the circle in D and C respectively. Prove that ABCD is a cyclic quadrilateral.



1. In order to prove that ABCQ is a cyclic quadrilateral, we need to show either one of the following conditions exists:
   a) One pair of opposite angles is supplementary **or**
   b) A line segment subtends equal angles at two points on the same side of it **or**
   c) An exterior angle is equal to its interior opposite angle

   The diagram and information suggests that we could probably use (c) i.e. An exterior angle is equal to its interior opposite angle

   The problem now reduces to finding a way of showing that <D1 = <B.

   How does one show these angles to be equal?

   Probably we can attempt to show that <D1 and < B are each equal to the same angle.
   Lets try!
   Can we relate <D1 to any other angle(s) in the given diagram.
   Yes, < D1 = < M2 (<s in the same segment)

   Can we show that <M2 = <B?
   If we can, then we may conclude that <D1 =<B and our proof will be done by using (c).

   Let us now see how can prove that <M2 = <B from information given.
   .
   PM is a diameter so <C3 = 90°, (< in semi –circle or < at centre theorem).
   Hence we can deduce that <MCB = 90° (PCB is a straight line)

Furthermore, <PMB = 90° (<between tangent and diameter/radius)
Hence we can deduce that <M2 + <M1 = 90°

Can we express <M2 in terms of <M1?
Yes, as follows: <M2 + <M1 = 90°
<M2 = 90° - <M1 …………………………………..(1)

Can we express <B in terms of < M1?
Yes, with the use of the sum of the <s of a triangle theorem.
In ΔMCB: < B + <MCB +<M1 = 180° .
Therefore <B + 90° + <M1 = 180°
Therefore <B = 90° - <M1 ………………………………… (2)

Using (1) & (2), we can conclude that <B = <M2 (why?)

So we now have successful plan to prove that ABCD is a cyclic quadrilateral.

Now we write out the proof in the opposite order to analytic thinking. We often use an analytic method to find a proof, but then we write out a synthetic proof because it is shorter and clearer.

Proof:
<C3 = 90° (< in semi-circle or < at centre = 2 X < at circumference)
[therefore] <MCB = 90° (PCB is a straight line)

and <PMB = 90° (<between tangent and radius/diameter)
therefore <M2 + <M1 = 90°
therefore <M2 = 90° - <M1 …………………………………(1)

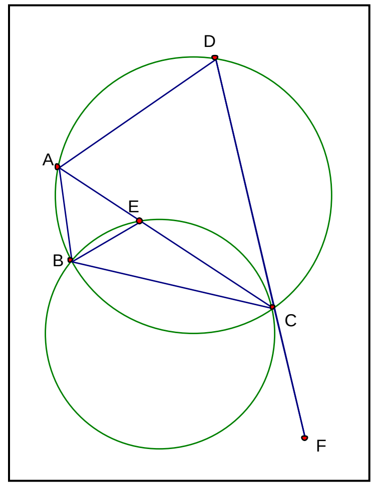
and < B + <MCB +<M1 = 180° (sum of <s of ΔPQB)
Therefore <B + 90° + <M1 = 180°
Therefore <B = 90° - <M1 ………………………………… (2)
But <M2 = 90° - <M1 (see (1))
Therefore <B = <M2
But <D1 = <M2 [<s in same segment]
Therefore <B = <D1
Therefore ABCD is a cyclic quadrilateral.

Note: This proof is synthetic. The analysis enabled us to develop the method.

We illustrate this analytic approach in the following example by means of a flow chart. It is not necessary to write down the analysis, but it its helpful if you jot down your ideas, perhaps in a working margin.

Example

In the accompanying figure ABCD is a cyclic quadrilateral. DC is a tangent to a circle through B and C, which cuts AC at E DC is produced to F. Prove that AD is a tangent at A to circle ABE.



Analysis

We have to prove that AD is a tangent, so try the converse of the tangent-chord theorem.

↓

Which angles must we show are equal?

↓

We need to prove that <DAC = <ABC?

↓

How can we prove that <DAC = <ABC?

↓

Express <DAC and <ABC in terms of other angles, which may have some possible links. Then use the following argument to prove that <DAC = <ABC: Quantities equal to the same quantity are equal to one another.

↓

How doe we start to express <DAC and <ABC in terms of other angles?

↓

Start by establishing which angles are equal from the given information. Then try to express <DAE and <ABE respectively , using the established equal angle(s) wherever possible.

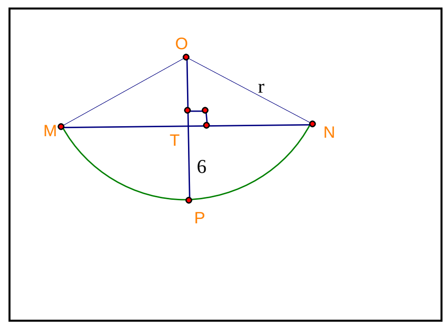
Note: The analysis enabled us to develop the method. We now write out the proof in the opposite order to the analytic thinking i.e we now write down the synthetic proof, which is shorter and clearer.

Proof:

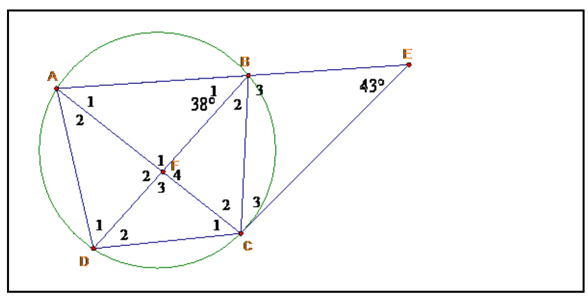
<BCF = <BEC (< between tangent and chord)
and <BCF = <BAD (ext. < of cyclic quad. ABCD)

Now, <DAE = <BAD - <BAE (trivial)
and <ABE = <BEC - <BAE (ext. < of ΔABE)
but<BEC = <BAD (both = <BCF)
Therefore <DAE = <ABE
Therefore AD is a tangent to circle centre ABE.

Exercise 9.11 (Mixed Exercise)

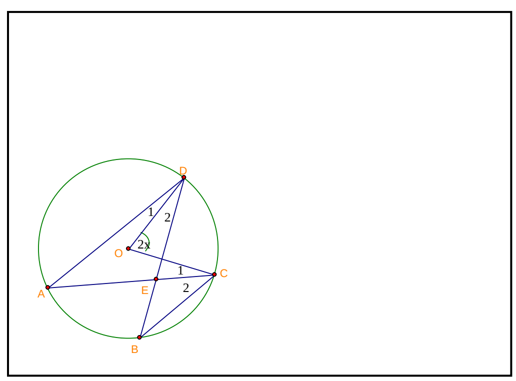


1.In the accompanying figure O is the centre of the circle of Which MPN is an arc. The point P lies on the arc MN such that OP is perpendicular to MN and intersects MN at T. The length of MN is 20 units , the length of TP is 6 units and the radius OM of the arc MPN is r units.
   (a)Express the length of OT in terms of r.
   (b)Why is AC = BC?
   (c)Find the value of r.



2. In the diagram alongside, ABCD is a cyclic quadrilateral such that BD bisects $\angle ADC$. AB is produced to E such that BE = BC. $\angle E = 43^\circ$ and $\angle B_1 = 38^\circ$. Calculate with reasons:

   (g) $\angle B_2$
   (h) $\angle D_2$
   (i) $\angle F_3$

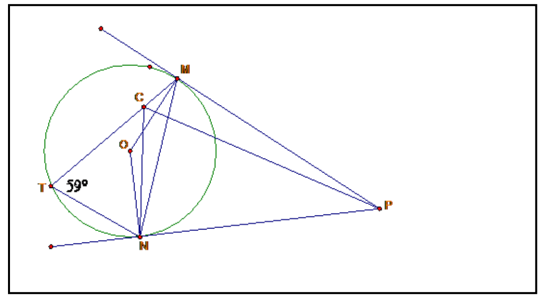


3. In the figure alongside, AD and BC are two parallel Chords of the circle with centre O. The chords BD and AC intersect at E and $\angle DOC = 2x$.

3.1. Name, giving reasons, three angles in the figure
Each of which is equal to x.

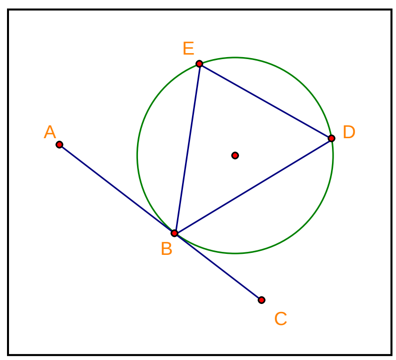
(a) Hence show that $\angle DEC = 2\angle A$
(b) Why are the points D,O, E and C concyclic.

4.In the accompanying figure, PM and PN are tangents to the circle with centre O. The point T lies on the major arc MN of the circle and C is a point on MT such that CT = CN. The size of $\angle T = 59^\circ$.

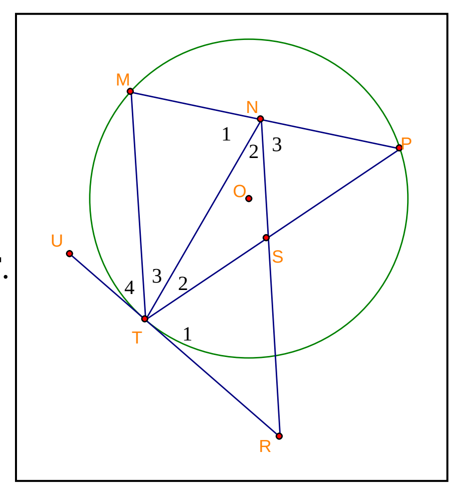


(a) What is the size of $\angle PMO$.
(Give a reason for your answer).
(b) Prove that PMON is a cyclic quadrilateral.
(c) Calculate with reasons, the size of
  (i) $\angle MON$
  (ii) $\angle MCN$
(d) Prove that MCNP is a cyclic quadrilateral.
(e) Hence, (or otherwise), show by calculation that PC bisects $\angle MCN$.

5. In the accompanying figure, ABC is a tangent to the circle at B. Copy the figure and use it to prove the theorem which states that $\angle CBD = \angle E$.
(Show all construction line in your diagram)



6.In the figure alongside, RTU is a tangent at T to the circle MTP. The straight line RSN is parallel to MN.
(c) Prove that $\angle CBD = \angle T_1$.
(d) Join P to R and state the type of quadrilateral PNTR is. Give a reason for your answer.
© If NM = NT, prove that RT is a tangent at T to the circle passing through the points N,S and T.



TRACKING MY PROGRESS

1. Answer these questions on your own. Then discuss in small groups. How do you feel about your ability to do the following:
   (a) Prove a rider or theorem using the direct application of theorem (s) approach.
   (b) Prove a rider or theorem using the congruency approach.
   (c) Prove a rider or theorem using the algebraic approach.
   (d) Prove a rider or theorem using the contradiction approach.
   (e) Prove a rider or theorem using the other branches of mathematics approach.
   (f) Prove a rider or theorem using the analytic approach.

   If you can't do any of these things, speak to our teacher or to anyone else who could help you.

2. Write down the three main steps that are essential in the form of proof that is described as proof by contradiction. Choose a rider or theorem in this chapter to illustrate how you would use this approach of proof by contradiction.
3. In an analytic proof we work backwards from what we have to prove. Thereafter we write out the proof in the opposite order to analytic thinking i.e. we actually write out a synthetic proof because it is shorter and clearer. Look at your work in this chapter and find a rider that required you to use the analytic approach. Make a copy of your analysis as well as of your final proof and place it in your portfolio.