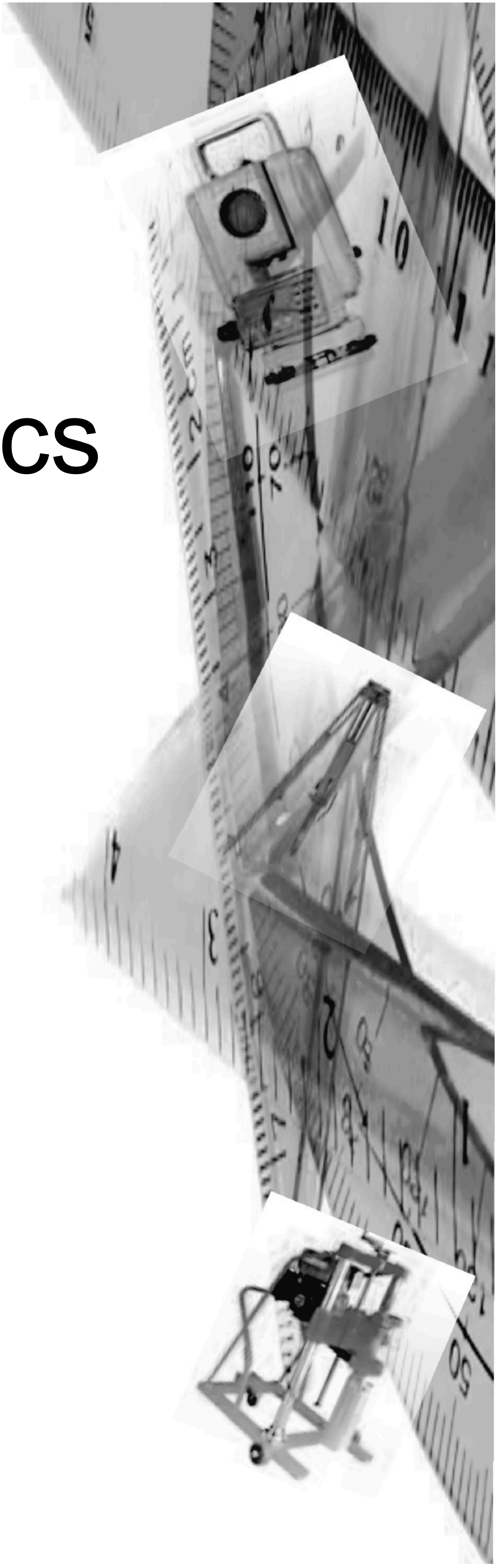
Mathematics



Unit 1

**Coordinate Geometry:
Points, Lines and Gradient**

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LESSON 1

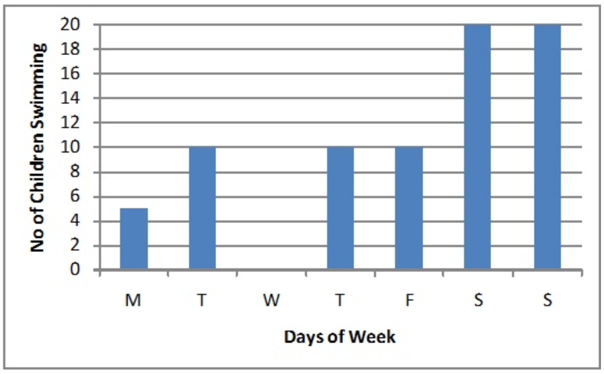
Coordinate Geometry: Points, Lines and Gradient

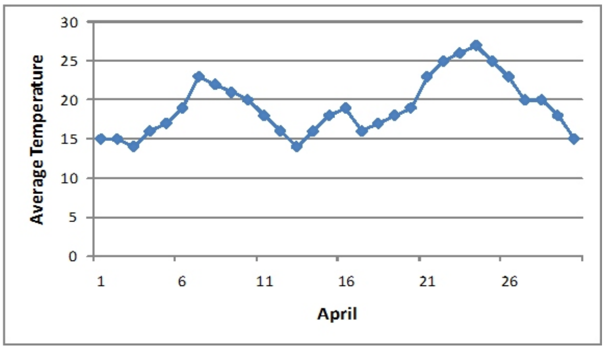
About this lesson

Coordinate geometry may seem like graphs to you but there are important differences between graphs and coordinate geometry. We use graphs to show information in pictures.

It is easy to see how one quantity changes in relation to another quantity if you represent the information on a graph. For example, we can draw a graph showing how many children swim in a pool on different days of the week. Or we can draw a graph showing the days of the month and the average temperature on each day.

Look at these examples:





Coordinate geometry has a far wider range of uses than graphs. Coordinate geometry shows the relationships between algebra and geometry. In coordinate geometry, we can express geometric ideas algebraically. We can also express algebraic ideas geometrically.

In this lesson you will:

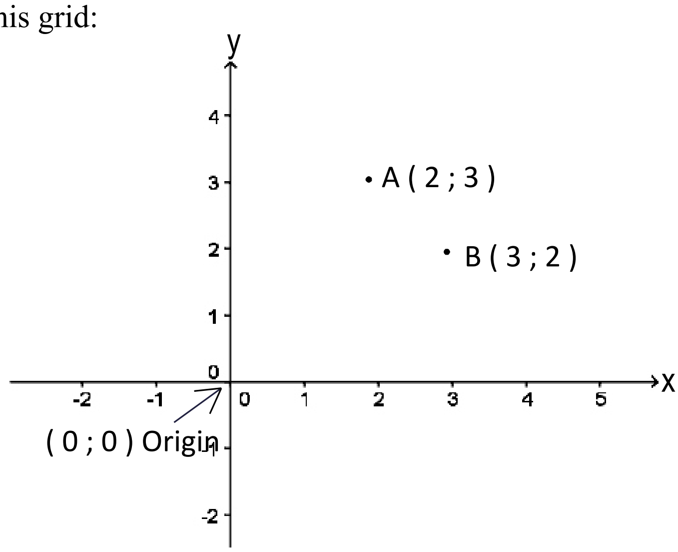
- plot points on a Cartesian plane
- name coordinates $(x; y)$
- understand the normal conventions and terms relating to axes
- draw lines such as $y = x$; $x = 3$; $y = 2$
- calculate and measure gradients
- relate gradients to the term collinear points
- calculate whether a point lies on a line or not

Coordinates of a point

In coordinate geometry, we define a point by a pair of numbers.

Example

Look at this grid:



The grid on which we plot the points is sometimes called the Cartesian plane. The word Cartesian comes from the name Descartes. He was a mathematician and a philosopher in the 17th century. He first introduced the idea that a point could be described using algebra.

The point at the centre of the cross is called the origin (0 ; 0). A on the grid is the point (2 ; 3). We get that by moving two steps to the right of the origin, and three steps up.

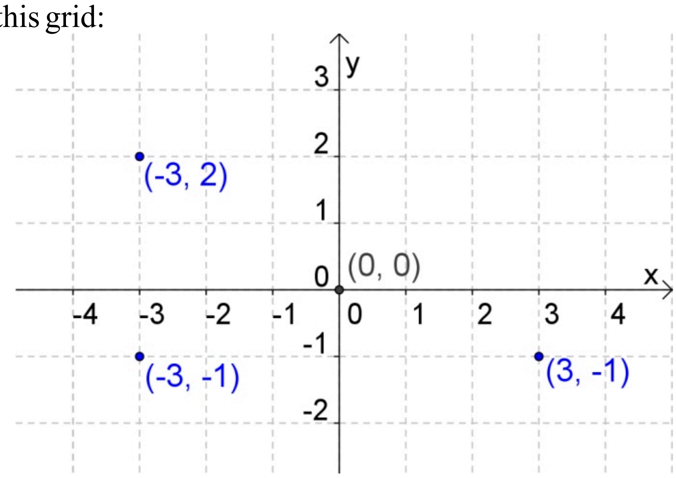
B is the point (3 ; 2). We get that by moving three steps to the right of the origin, and then two steps up.

Note that the order is very important. The point (2 ; 3) is called an ordered pair. We call the 2 and the 3 the coordinates of the point (2 ; 3). The first number is the x –coordinate (the horizontal distance across from the origin), and the second number is the y –coordinate (the vertical distance up or down from the origin).
The horizontal line we call the x –axis. The vertical line we call the y –axis.

$(2\,;3)$
coordinates
$(x\,;y)$

The x -coordinate describes horizontal movement along the x -axis and the y -coordinate describes vertical movement along the y -axis.

Look at this grid:

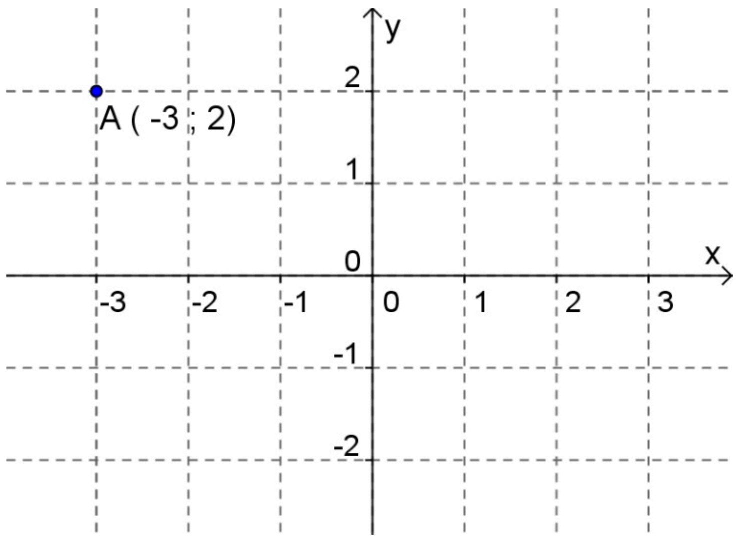


➔ C is the point (–3 ; 2). Start at the origin (0 ; 0). Move three steps to the left of the origin, along the x –axis, and then two steps up, parallel to y –axis.

➔ D is the point (–3 ; –1). Start at the origin. Move three steps to the left of the origin, along the x –axis, and then one step down, parallel to the y –axis.

➔ E is the point (3 ; –1). Start at the origin. Move three steps to the right of the origin, along the x –axis, and then one step down, parallel to the y –axis.

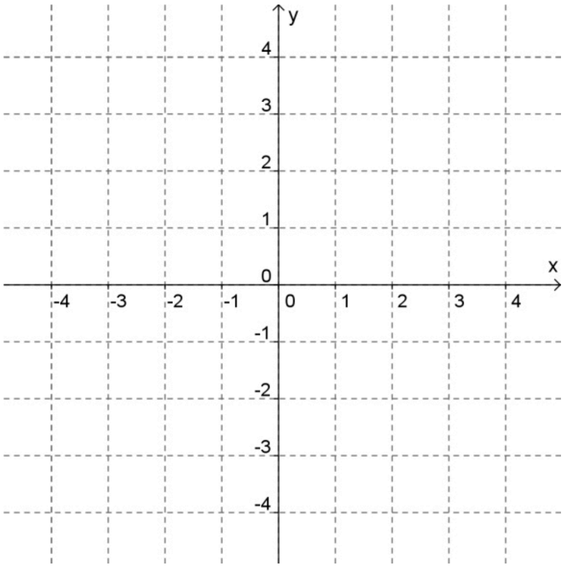
Let us try. Start at the origin. Move two steps up along the y –axis, and then three steps to the left, parallel to the x –axis . Yes, it is the same point.



ACTIVITY 1

Plot the following points on the grid. Label each point.

1. $A(1\,;3), B(3\,;1)$
2. $C(-1\,;-3), D(-3\,;-1)$
3. $E(-3\,;-1), F(-1\,;-3)$
4. $G(1\,;-3), H(3\,;-1)$
5. $I(0\,;3), J(-3\,;0)$



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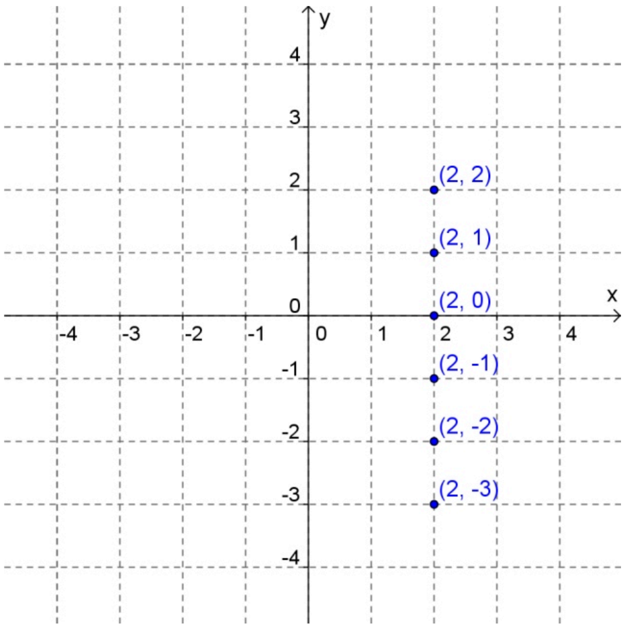
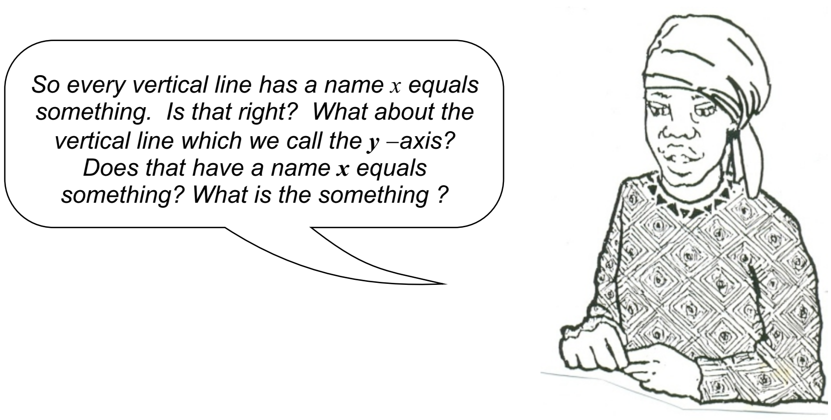
Naming lines

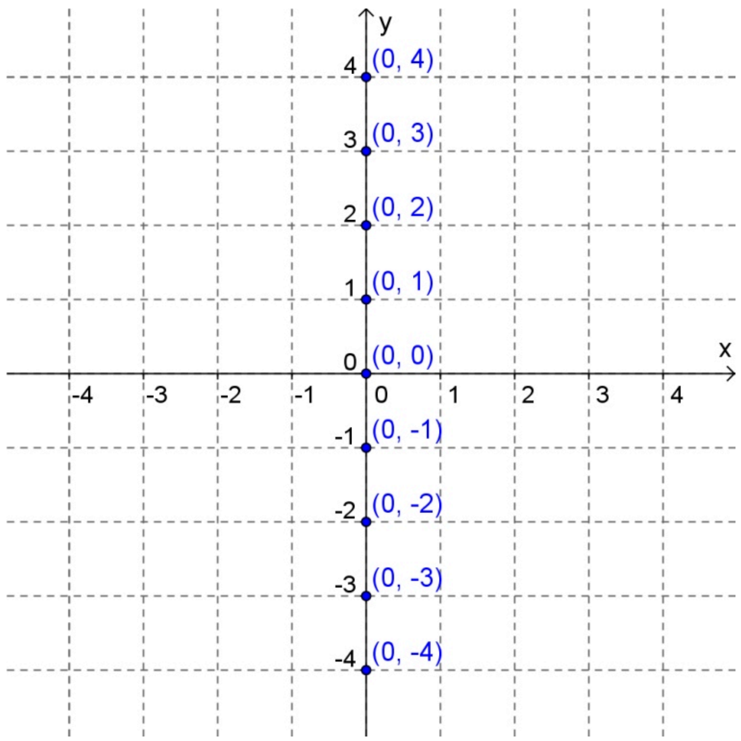
Look at this grid.
What do you notice about the points on the grid?



Firstly, the points are in a straight vertical line. Secondly the x –value is the same for each of the points and this is the line $x = 2$. Similarly the line $x = 3$, is the vertical line along which all the x –values are 3. The line $x = -2$ is the vertical line along which all the x –values equal -2.



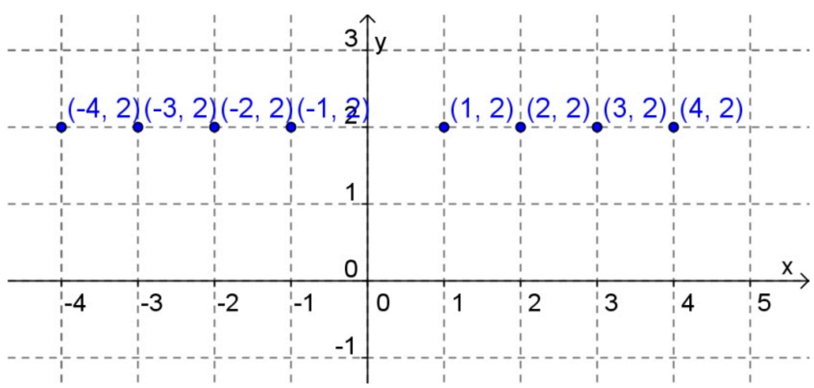
Well, let's have a look at that. Plot and name all the points along the y –axis .



We can give the y –axis the name $x = 0$, because at every point along the y –axis, the value of the x –coordinate is 0.

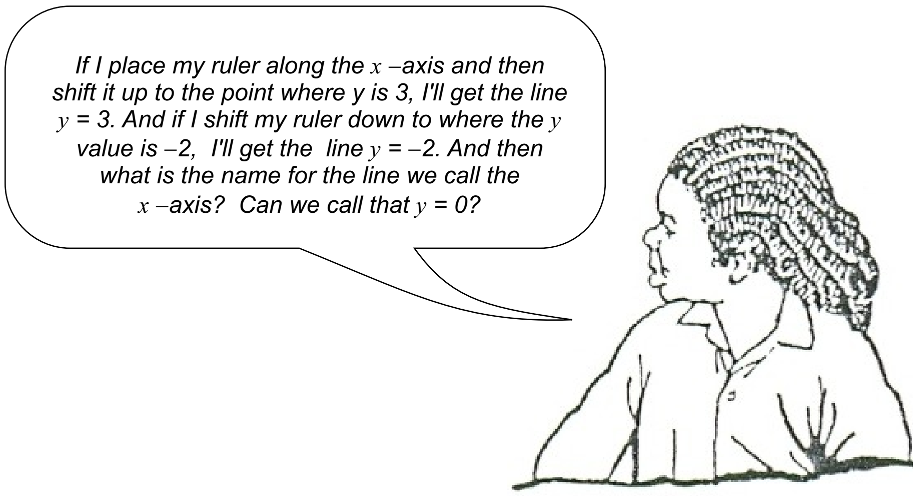
Example

Look at this grid.
What do you notice about the points?



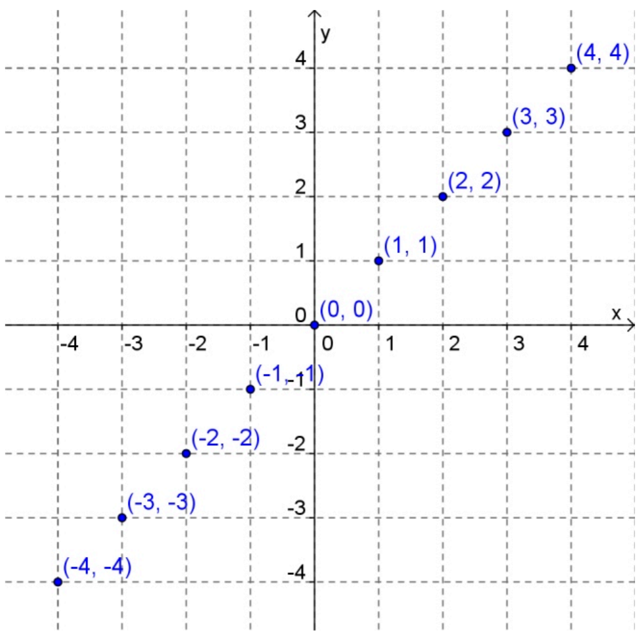
Solution

This time they are in a straight horizontal line. The the y –coordinate (the second of the pair) is the same in each case, and is always equal to 2. We call this line $y = 2$.



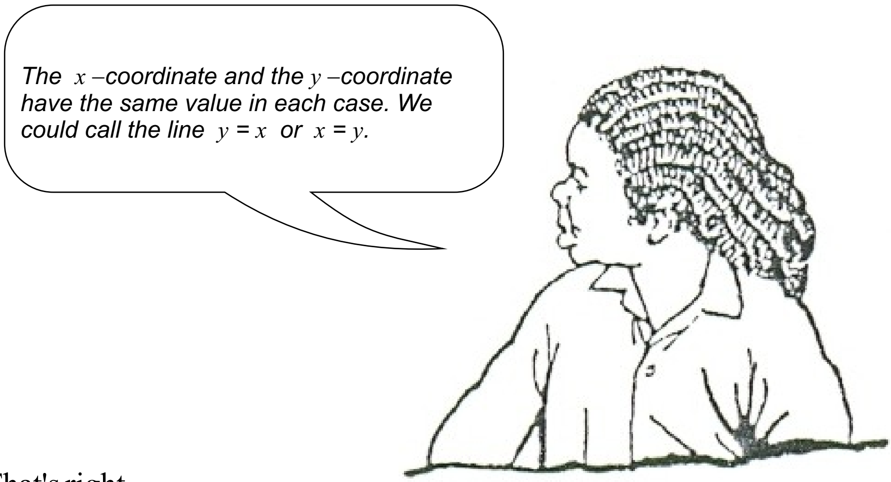
Yes, we can, because every y –value along the x –axis is equal to 0. Now we have named all the vertical lines and all the horizontal lines.

Well, have a look at the following points.



What do you notice about all the points plotted on this grid?

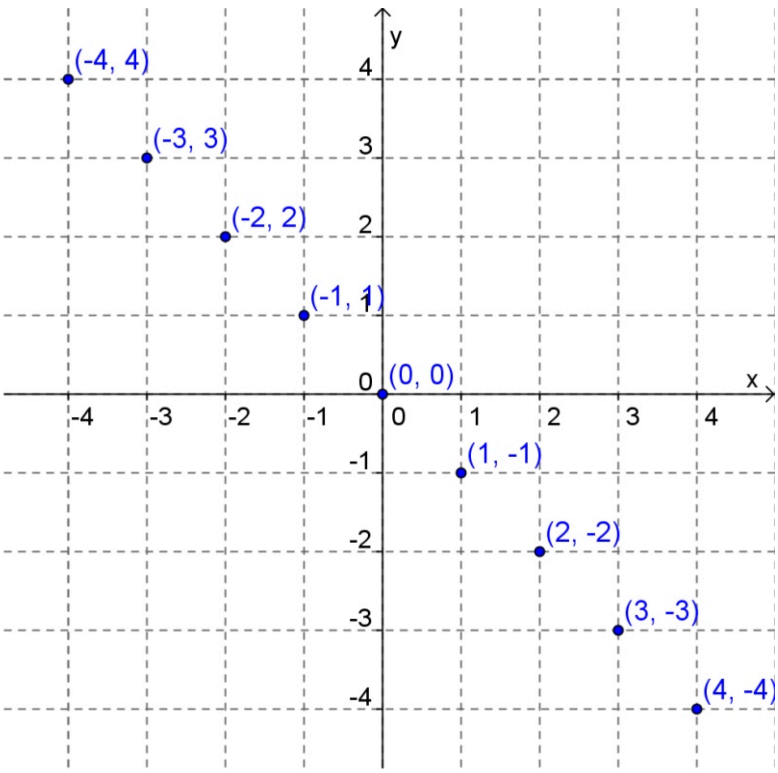
They are in a straight line. But the line is not horizontal or vertical. It is called a diagonal. Do you notice anything else about those points?



That's right.

Let's plot the points along the other diagonal and see if we can find a name for that line.

Look at this grid.



What do you notice about the points on that diagonal?

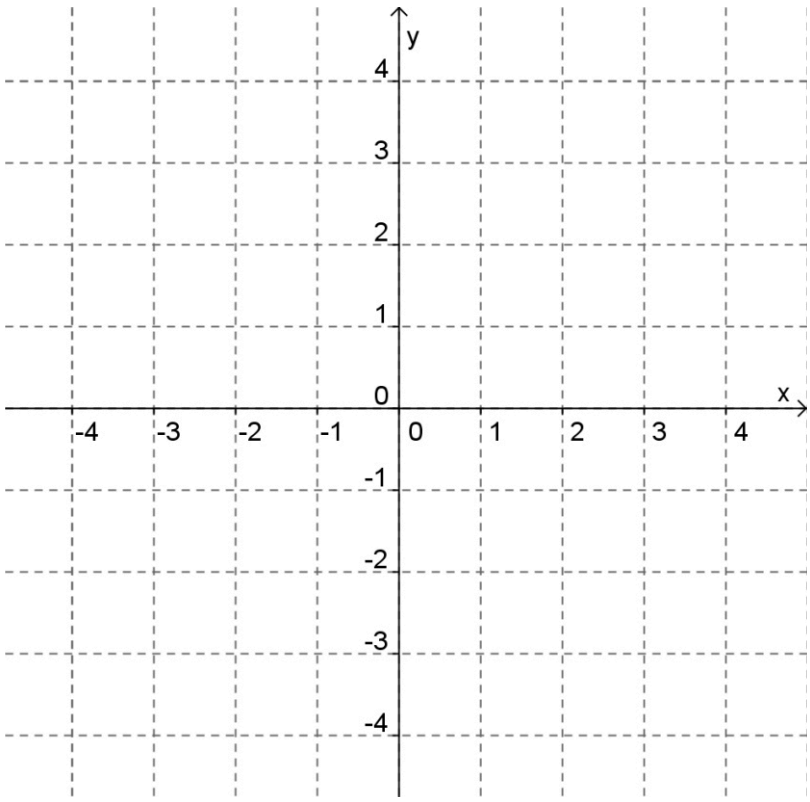
In each case the x –value is equal to the y –value but with the opposite sign. We call that line $y = -x$.

ACTIVITY 2

1. Draw and label the following lines on grid.

   a) $y = 2$ b) $x = -3$ c) $x = 1$ d) $y = -1$

2. Draw and label the lines $y = 2x$ and $y = -2x$ on a new grid.



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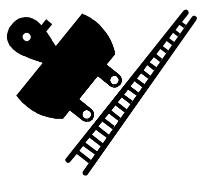
The gradient of a line



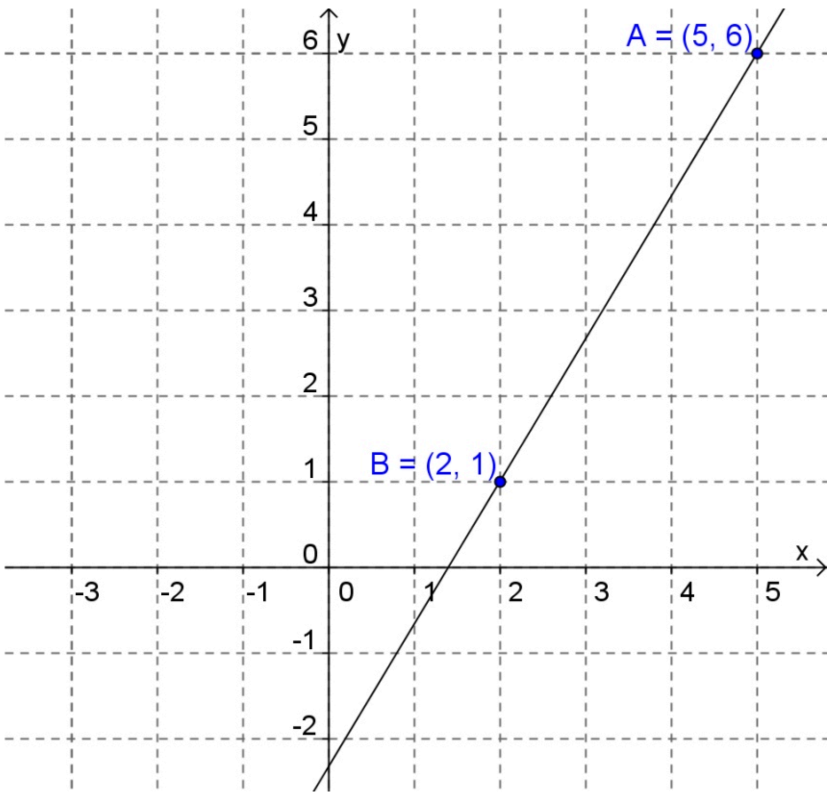
The gradient or slope of a line between two points is found using this formula:

$$m = \frac{\text{the increase in the } y\text{ - values}}{\text{the increase in the } x\text{ - values}}$$

$$= \frac{\text{change in } y\text{ - value}}{\text{change in } x\text{ - value}}$$

For example, take the straight line below. The change in the y –value between A and B is +5. The change in the x –value between A and B is +3.

So the gradient of AB is $\frac{5}{3}$.



If you think of going from A to B, the gradient is $\frac{-5}{-3} = \frac{5}{3}$.

So, it doesn't matter whether you think of going from A to B or from B to A, as long as you move in the same direction for both the x –value and the y –value.

More formally, the formula may be remembered as: $m = \frac{y_2 - y_1}{x_2 - x_1}$
So how do you use this?

Example
Let's use the previous example of $A(2;1)$ and $B(5;6)$.

$$A(2;\ 1) \qquad B(5;\ 6)$$
$$\downarrow\ \downarrow \qquad\quad \downarrow\ \downarrow$$
$$(x_1; y_1) \qquad (x_2; y_2)$$

The formula for gradient is $m = \frac{y_2 - y_1}{x_2 - x_1}$ so gradient from A to B is found by substituting 6 for y_2 and 1 for y_1 , 5 for x_2 and 2 for x_1 .

Therefore $m = \frac{6-1}{5-2} = \frac{5}{3}$

(5;6) *is* $(x_1; y_1)$ *or* $(x_2; y_2)$ *but not* $(x_1; y_2)$.

Good question. Let's take a look.

$$B(5;\ 6) \qquad A(2;\ 1)$$
$$\downarrow\ \downarrow \qquad \downarrow\ \downarrow$$
$$(x_1; y_1) \qquad (x_2; y_2)$$

$$\therefore\ m = \frac{1-(+6)}{2-(+5)} = \frac{1-6}{2-5} = \frac{5}{3}$$ The answer is the same.

So you see that the gradient does not change according to the order of numbering.

It is important to practice using this concept so try Activity 3.

ACTIVITY 3

Find the gradients between the points in each of the following examples:

1. (1;3), (2;7)
2. (–2;3), (3;–2)
3. (1;–3), (5;–3)
4. (–4;3), (–3;–4)
5. (2;3), (2;7)

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Let's move on to the next topic and find out about points in a straight line.

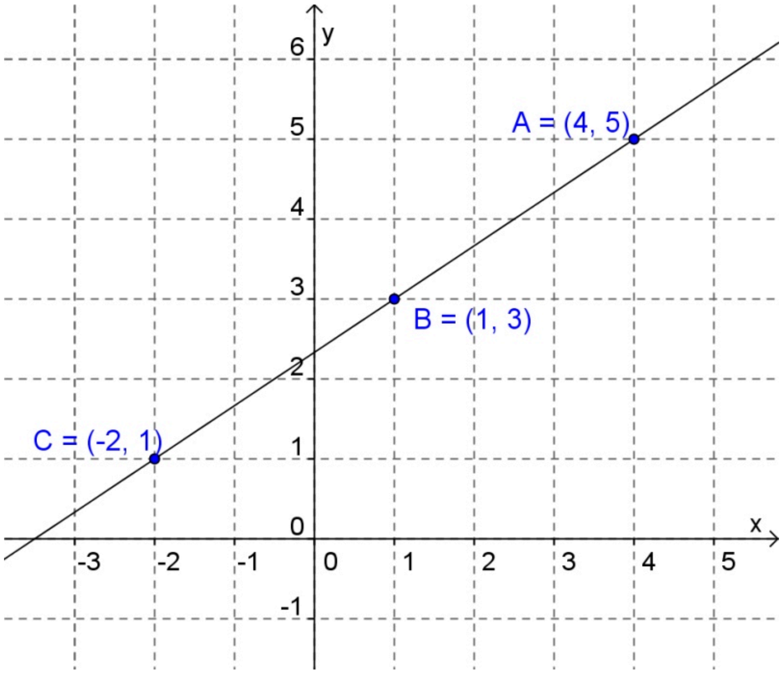
Collinear points

If points are in a straight line we say they are collinear. Let's look at an example.

Example

Are $A(4;5)$; $B(1;3)$ and $C(-2;1)$ on the same line?

Solution
Well, let's take a look.



On the graph they look as if they are in the same line but just 'looking' is not acceptable mathematically, so you have to prove it. This is not as difficult as you may think. If you find the gradient of AB and then the gradient of BC and the two gradients are equal, then the points A, B and C are in a straight line.

$\therefore$ therefore
$=$ equal to
$\neq$ not equal to

We have discussed the problem so let's do the mathematics.

If $A(4;5)$ and $B(1;3)$
$(x_1;y_1)$ $(x_2;y_2)$

The gradient of AB = $\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{3-5}{1-4}=\dfrac{-2}{-3}=\dfrac{2}{3}$

Now look at $B(1;3)$ and $C(-2;1)$

The gradient of BC = $\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{1-3}{-2-1}=\dfrac{-2}{-3}=\dfrac{2}{3}$

The gradient of AB is equal to the gradient of BC so the points A, B and C must lie on the same line.

Alternatively, if the gradient of AB $\neq$ BC, then the points do not lie on the same line.

You will need to practice this, so try Activity 4.

ACTIVITY 4

In each of the following cases, decide whether the given points are on a straight line (collinear).

1. $A(-2;0)$, $B(1;3)$ and $C(0;2)$

2. $P(4;6)$, $Q(8;12)$ and $R(12;18)$

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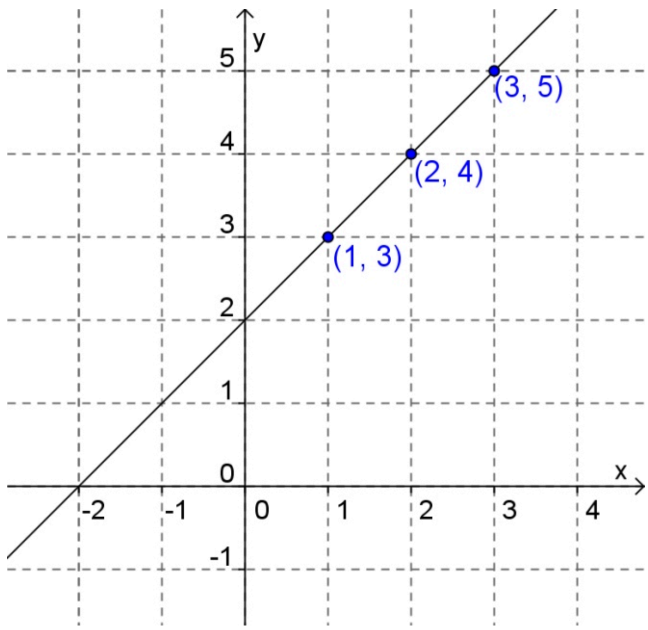
3. $L(-5;3)$, $M(7;10)$ and $N(9;14)$

Points on a line

If we plot the points (1;3), (2;4), (3;5), we would get a straight line.
$y = x + 2$ is the equation of the line.
Do you notice that each time the y –coordinate is 2 more than the x –coordinate?



Yes, that's right, and this was the breakthrough that Descartes made in the 17th century. Every line can be represented by an equation and every equation can be drawn as a line.

Now, there is another interesting thing about coordinate geometry. We can work out whether a point is on the line or not.

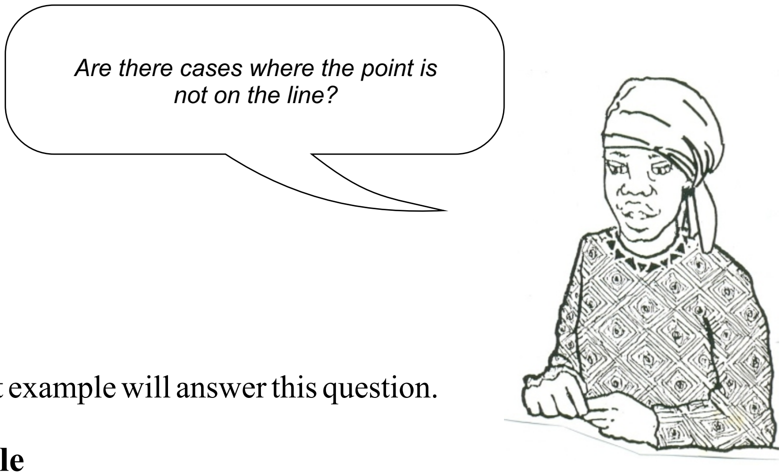
Example
Does the point (4; 6) lie on the line $y = x + 2$?

Solution
We substitute the x –coordinate in the place of x in the equation and then solve for y. If the y –value that results is the same as the y –value of the point, it follows that the point (4;6) is on the line.

LHS = Left hand side
RHS = Right hand side

Let's try.
We substitute 4 in place of x $\therefore$ $y = 4 + 2 = 6$.

Since the y –value is also 6, we can say that the point (4;6) lies on the line.



Our next example will answer this question.

Example
Does the point (–2;3) lie on the line $y = x + 2$?

Solution
Let's take a look.

$y = (-2) + 2 = 0$

Since the y –value of the point is **not** 0, we can say that the point (–2;3) does **not** lie on the line.

Check this by finding (–2;3) on the grid on the previous page. You will see that it is not on the $y = x + 2$ line.

ACTIVITY 5

In each of the following cases decide whether the given points are on the given lines.

1. $y = 2x + 3$ and the point is (0;3)
2. $y = 4x + 1$ and the point is $(\frac{1}{4};0)$
3. $y = x^2 + x + 2$ and the point is $(-2;8)$

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Summary

In this lesson you learnt that:

- a point is named by the coordinates $(x; y)$
- the grid on which we plot the points is sometimes called the Cartesian plane
- the x - axis is the horizontal axis, the y –axis is the vertical axis
- the gradient of a line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$ where $(x_1; y_1)$ and
- $(x_2; y_2)$ are any two points on the line
- points that lie on the same line are collinear and have the same gradient between them
- it is possible to calculate whether a point lies on a line or not.

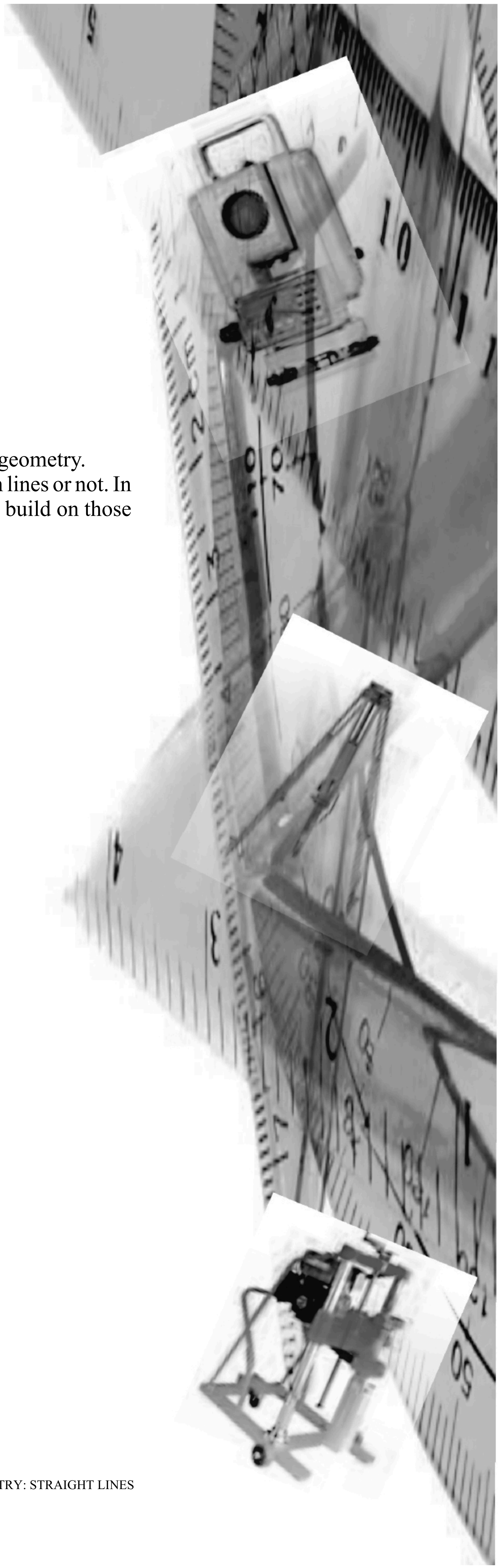
CHECKLIST

Are you able to:

- plot points on a Cartesian plane
- name coordinates $(x;y)$
- understand the normal conventions and terms relating to axes
- draw lines such as $y = x; x = 3; y = 2$
- calculate and measure gradients
- relate gradients to the term collinear points
- draw lines such as $y = x$, $x = 3$ and $y = 2$?

LESSON 2

Coordinate Geometry: Straight Lines



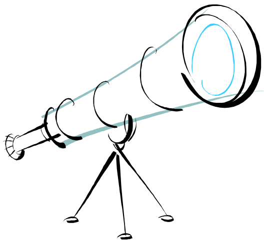
About this lesson

In Lesson 1 you looked at most of the basics of coordinate geometry. You learnt about collinear points and whether points lie on lines or not. In this lesson you will revise some of the earlier principles, build on those and learn some new ones.

In this lesson you will:

- draw lines given in standard form
- find the equation of a line
- calculate if lines are parallel
- calculate if lines are perpendicular
- calculate whether a point lies on a line or not

The equation of a line



It is possible with coordinate geometry to find the equation of any straight line given a point on the line and the gradient of the line. An astronomer might need to follow the path of a star. It is possible for the astronomer to find an equation for that path and to make calculations on paper which would be correct. The wonder of mathematics is that you can work with what you know and then apply your knowledge to the unknown.

Example

Find the equation of the straight line with a gradient of 3, and which passes through the point (2;–1).

Let us revise the method you learnt previously. We know that we can find the gradient if we have two sets of points. In this problem we have only one point (2;–1). Let's call it $(x_1; y_1)$ and the gradient $m = 3$.

Solution

Any other point on the line can be $(x_2; y_2)$.

$m = 3,\ x_1 = 2$ and $y_1 = -1$

Since $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $y_2 - y_1 = m(x_2 - x_1)$

$y-(-1) = 3(x-2)$ (Substitute $x_1 = 2$ and $y_1 = -1$)

$\therefore y + 1 = 3x - 6$ (Expand the right hand side)

$\therefore y = 3x - 7$ (Simplify the equation)

ACTIVITY 1

*These require some thinking!

In each of the following cases, give the equation of the straight line, with the given gradient and passing through the given point.

1. $(-2; 4), m = 2$

2. $(-2; -3), m = 1$

3. $(3; -5), m = -1$

4.* $(-5; -2), m = 0$

5.* $(-3; 2)$, m is meaningless because the line is vertical.

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You can find the equation of a straight line if you know any two points on the line.

Example

Find the equation of the straight line through two points $A(2\,;3)$ and $B(1;9)$.

Solution
It is a slight variation of the previous method.

We know that $y_2 - y_1 = m(x_2 - x_1)$ but in this case we are not given m so we need to work it out first.

Let's consider $A(2\,;3)$ to be $(x_1; y_1)$ and $B(1;9)$ to be $(x_2; y_2)$

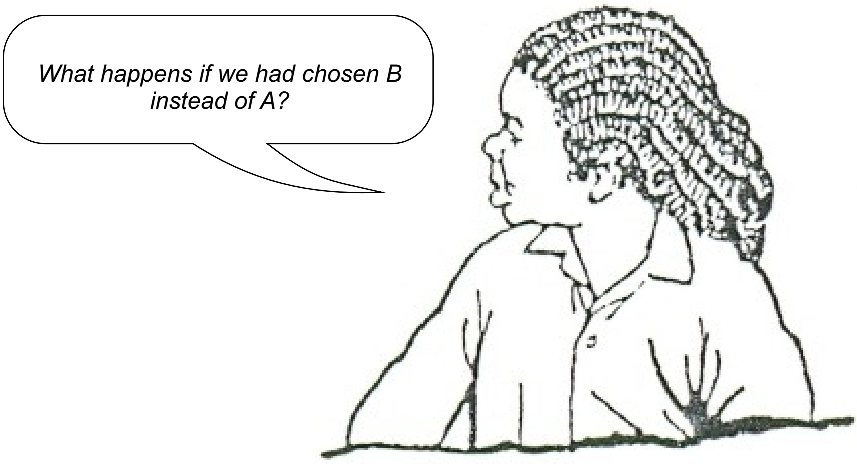
Then $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ so $m = \dfrac{(9-3)}{(1-2)} = \dfrac{6}{-1} = -6$

Now that we know that $m = -6$, we can use the equation $y_2 - y_1 = m(x_2 - x_1)$. We only need to substitute 1 point, so we choose either A or B.

Let's use $A(2\,;3)$.

$m = -6,\ x_1 = 2$ and $y_1 = 3$

$y - 3 = -6(x - 2)$ (Substitute $x_1 = 2$ and $y_1 = 3$)
$\therefore\ y - 3 = -6x + 12$ (Expand the right hand side)
$\therefore\ y = -6x + 15$ (Simplify the equation)



Let's check. We'll use $B(1;9)$ instead of A.

$m = -6,\ x_1 = 1$ and $y_1 = 9$

$y - 9 = -6(x - 1)$ (Substitute $x_1 = 2$ and $y_1 = 3$)
$\therefore y - 9 = -6x + 6$ (Expand the right hand side)
$\therefore y = -6x + 15$ (Simplify the equation)

We can see that we get the same equation, so it doesn't matter which point we use. It may be a good idea to choose a point and then use the other point to check your answer.

ACTIVITY 2

Find the equations of the straight lines through the following points:

1. (3; –5), (–1; 3)
2. (4; 7), (–2; 2)
3. (–2; 3), (1; 3)
4. (–1; 2), (–1; 5)

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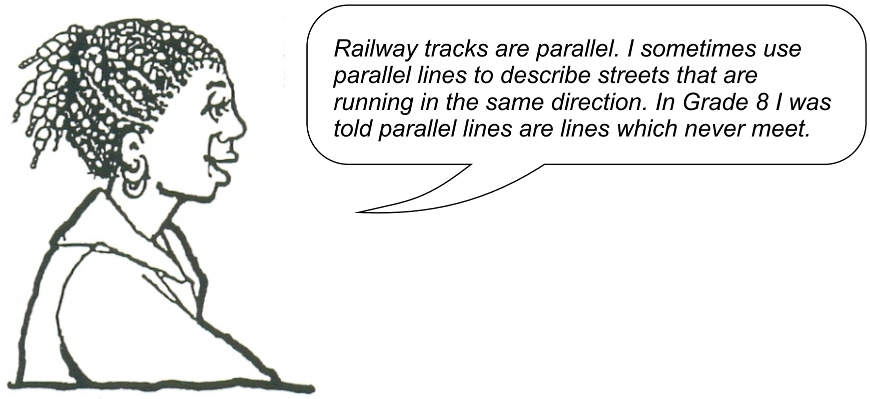
Parallel lines

Can you describe or define parallel lines?

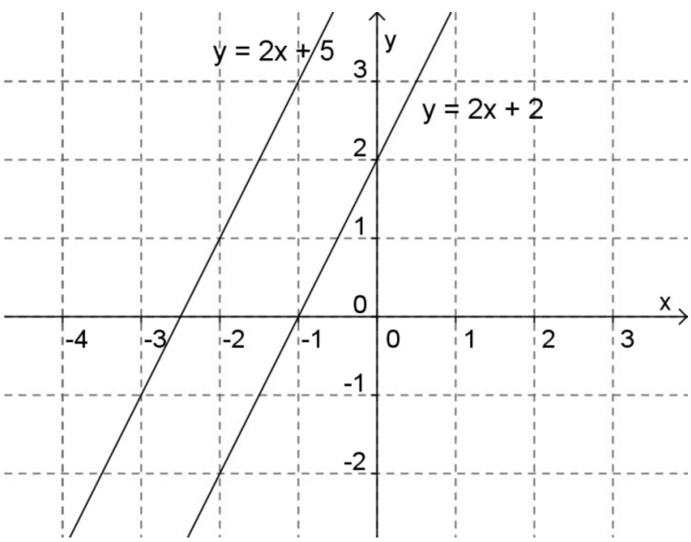


Those answers are correct in the contexts in which you used them. However, in coordinate geometry we need to have a clearer definition.

Two lines are parallel if their gradients are equal.

Example

Draw the lines of $y = 2x + 5$ and $y = 2x + 2$ What do you notice?



Solution
The two lines are parallel and they have the same gradient (slope) of 2. You will also notice that the second line is 3 units above the first for every value of x.

ACTIVITY 3

Sketch the graphs of the following on the same set of axes and say what you notice.

a) $y = 3x$
b) $y = 3x + 1$
c) $y = 3x - 3$

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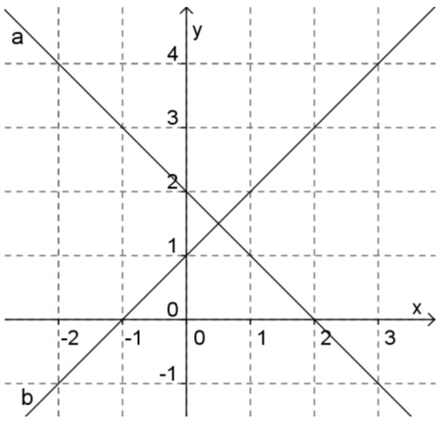
Perpendicular lines

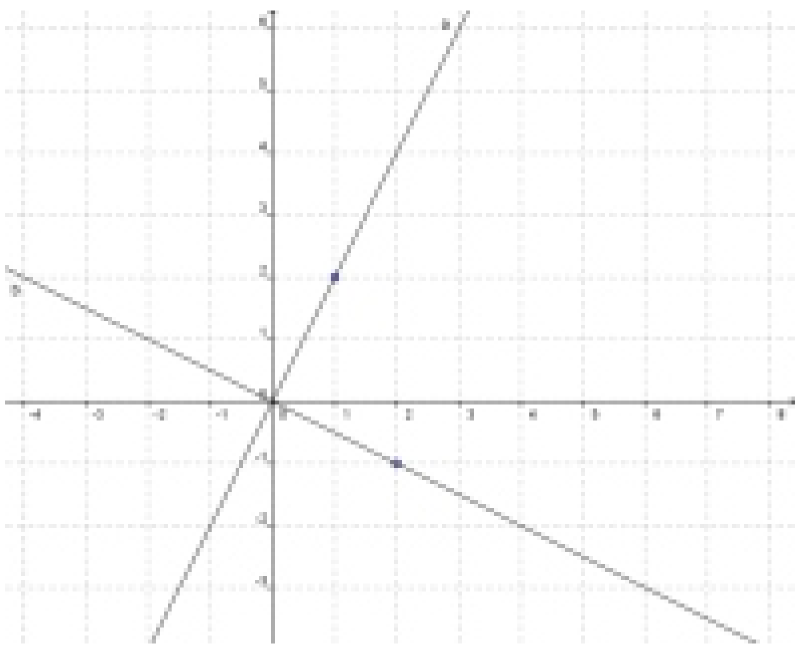
If two lines are perpendicular to one another it means they are at right angles to each other. Try the activity below to find out more about perpendicular lines.

Example
Find the gradients of lines a and b in each sketch.

1.



2.

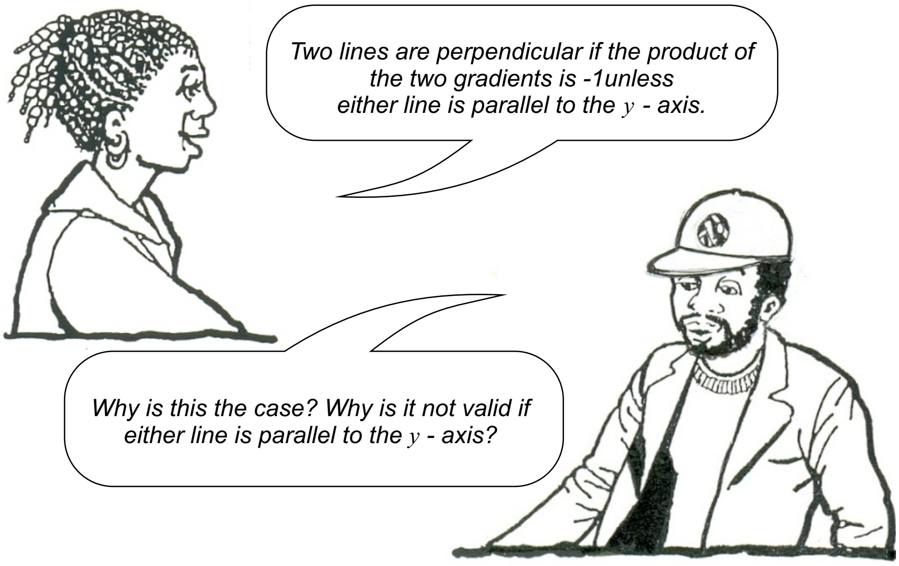


Solution

1. The gradient of $a = -1$ and the gradient of $b = 1$.

2. The gradient of $a = 2$ and the gradient of b is $-\frac{1}{2}$.

In each case the product of the gradients is -1.



The slope of a line which is parallel to the y –axis (vertical line) is undefined i.e. there is no gradient and m is undefined. So there is nothing that can be multiplied with the gradient of the other line.

Also, the line perpendicular to a vertical line would be a horizontal line. The slope of a horizontal line is zero. There is no number, which when multiplied with zero, could give an answer of -1.

But obviously vertical lines are perpendicular to horizontal lines.

Example

Determine whether the following pairs of lines are parallel or perpendicular or neither. Remember that each line should be in standard form ($y = mx + c$) before you begin.

1. $y = 2x + 2$ and $y - 2x = 3$

2. $y = 2x + 2$ and $y = -2x + 3$

3. $y = 2x + 2$ and $2y + x = 6$

Solution

1. You will need to rewrite the second equation in standard form so the question changes to $y = 2x + 2$ and $y = 2x + 3$.
   The gradient in both is equal to 2, so the lines are parallel.

2. You do not have to rewrite these

$y = 2x + 2$ and $y = -2x + 3$

$\therefore m = 2$ and $m = -2$

The gradients are not the same and so the lines are not parallel.

$2 \times (-2) = -4 \neq -1$

The product of their gradients is not -1 and so the lines are not perpendicular either.

3. You need to rewrite the second equation so the question changes to

$y = 2x + 2$ and $2y = -x + 6$

$\therefore y = -\frac{1}{2}x + 3$

$\therefore m = 2$ and $m = -\frac{1}{2}$

The gradients are not the same and so the lines are not parallel.

$2 \times (-\frac{1}{2}) = -1$

$\therefore$ the lines are perpendicular.

ACTIVITY 4

1. Calculate the gradients of the lines joining the points:

   a) $(-3; 2), (1; 1)$
   b) $(4; 3), (-1; 8)$
   c) $(-3; -5), (1; 3)$
   d) $(a; b), (b; a)$

2. Write down the gradient of the line perpendicular to each of the lines in question 1.

3. Calculate the gradients of AB and CD in each case and state whether the lines are parallel, perpendicular or neither parallel nor perpendicular.

   a) $A(0; 1), B(-4; -2), C(-3; -1), D(1; 2)$
   b) $A(6; -10), B(0; 4), C(3; 0), D(-4; -3)$
   c) $A(-2; -4), B(3; 1), C(-5; -1), D(0; 3)$

ANSWERS ON PAGE 90

CHECKLIST

Are you able to:

- calculate the equation of a straight line given the gradient and a point on the line
- calculate the equation of a straight line given two points on the line
- decide from the equations of two straight lines whether the lines are parallel or perpendicular or neither.

Now try the self-check exercise that follows. If you can answer all the questions correctly then move on to the next lesson. If you have problems with any of the questions, go back to that section and re-read the explanation and examples. Then try again.

SELF-CHECK EXERCISE

1. Sketch graphs of the following, each on a separate grid.

   a) $y = 2x - 4$
   b) $y = -2$
   c) $x = 1$
   d) $3x + 2y - 6 = 0$

2. Determine p in each case if

   a) $y = 3x + 2$ is parallel to $y = px + 3$
   b) $y = 3x + 2$ is perpendicular to $y = px + 3$
   c) $y = -x + 4$ is parallel to $y = px + 2$
   d) $y = -x + 4$ is perpendicular to $y = px + 2$

3. Find the equations of the following straight lines:

   a) passing through the point (3; –2) and parallel to $y = 3x + 1$.
   b) passing through the point (3; –2) and perpendicular to $y = 3x + 1$.

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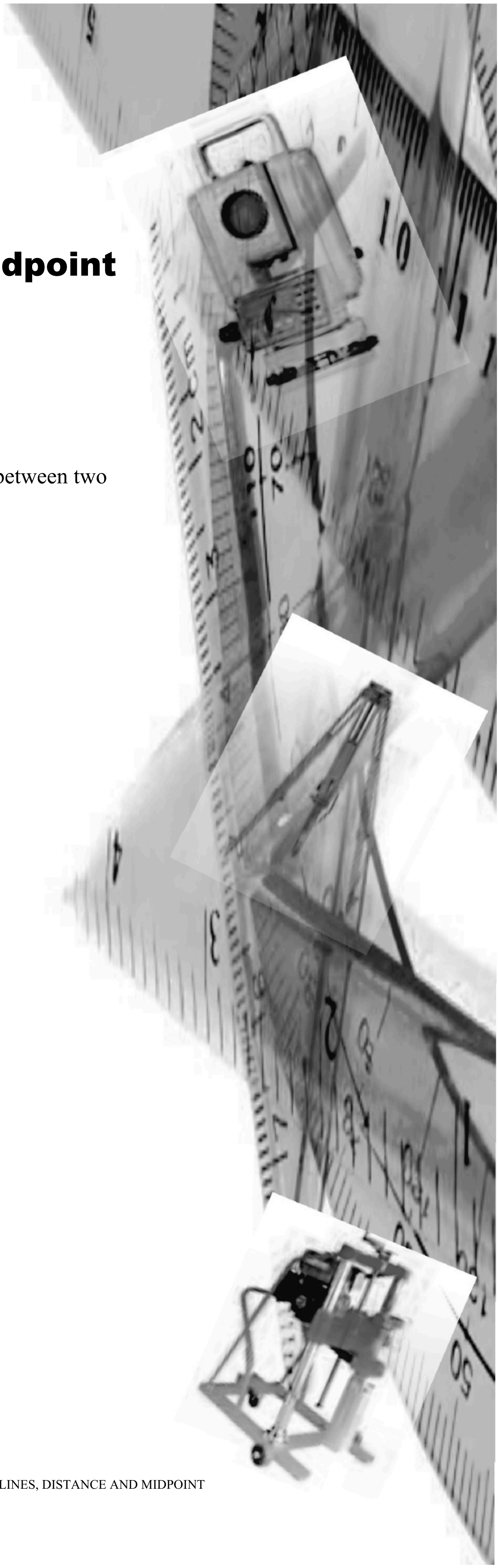
4. a) Does the point (2; 4) lie on the straight line $2x - y = 0$?
   b) Are the points (4; 5), (2; –1) and (0; –3) collinear or not?

LESSON 3

Coordinate Geometry: More lines, distance and midpoint



About this lesson

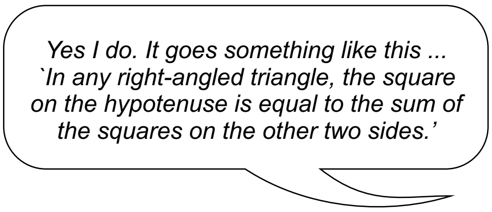
In coordinate geometry we are able to find the distance between two points. Let's have a look and see if we can work this out.

In this lesson you will:

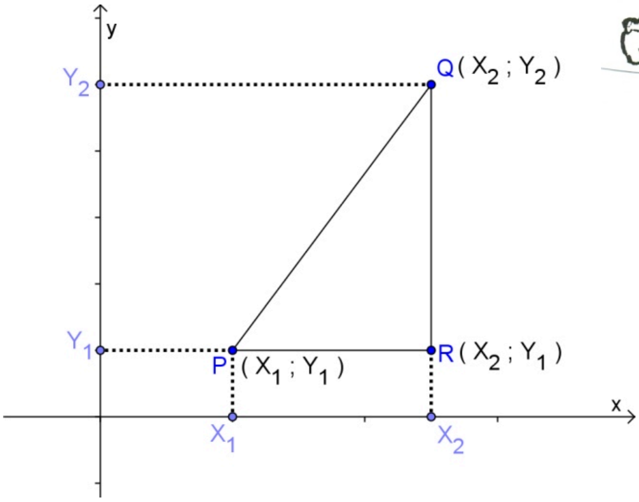
- find the distance between two points
- calculate the midpoint of a line segment
- find the point of intersection of two lines

The distance between two points

Descartes found in the 17th century that by bringing algebra and geometry together, all kinds of problems could be solved. In finding the distance between two points, we are going to use the Theorem of Pythagoras. Do you remember it?



Yes, that is right, but let's draw a picture to make it absolutely clear.



To calculate the distance from P to Q, we need to know the lengths of PR and RQ.

Firstly call $P(x_1; y_1)$ and $Q(x_2; y_2)$. Therefore the distance from P to R is $x_2 - x_1$ and from R to Q is a distance of $y_2 - y_1$.

We can say that $PQ^2 = PR^2 + RQ^2$ (Pythagoras' theorem)

$\therefore\ PQ^2 = (x_2 - x_1)^2 + (y_2 - y_2)^2$

$\therefore\ PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_2)^2}$

It is easier to remember this in the form:

The distance between two points $= \sqrt{(change\ in\ x)^2 + (change\ in\ y)^2}$

Distance between two points $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_2)^2}$

Example
Find the distance between (3; 2) and (5; 2).

Solution

Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_2)^2}$

Let (3; 2) be $(x_1; y_1)$ and (5; 2) be $(x_2; y_2)$

$$d = \sqrt{(5-3)^2 + (2-2)^2}$$
$$= \sqrt{4-0}$$
$$= 2$$

ACTIVITY 1

Find the distances between the following points:

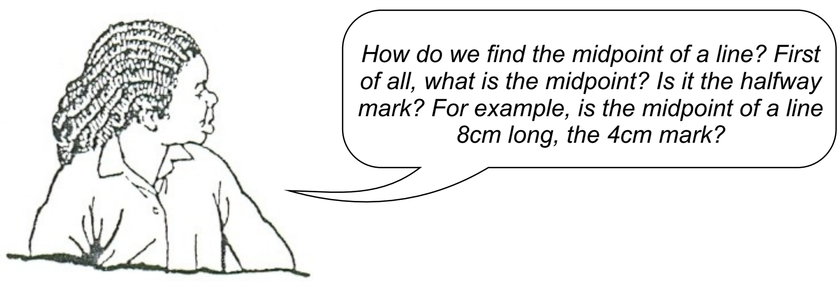
1. (6;–2) and (3;-6)

2. (2; 7) and (0; 2)

3. (0; 3) and (4; 0)

ANSWERS ON PAGE 91

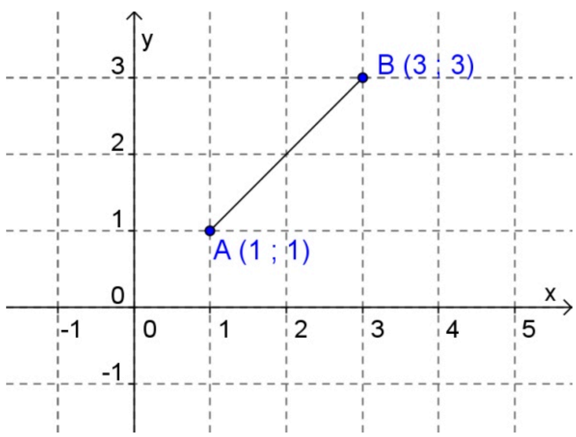
The midpoint of a line segment



Yes, that's right. That is how we understand things in everyday life, but let's look at what it means in coordinate geometry.

Let us start by taking a line segment we know. We make A the point (1; 1) and B the point (3; 3).

Look at this diagram. It is easy to see that the midpoint is (2; 2).

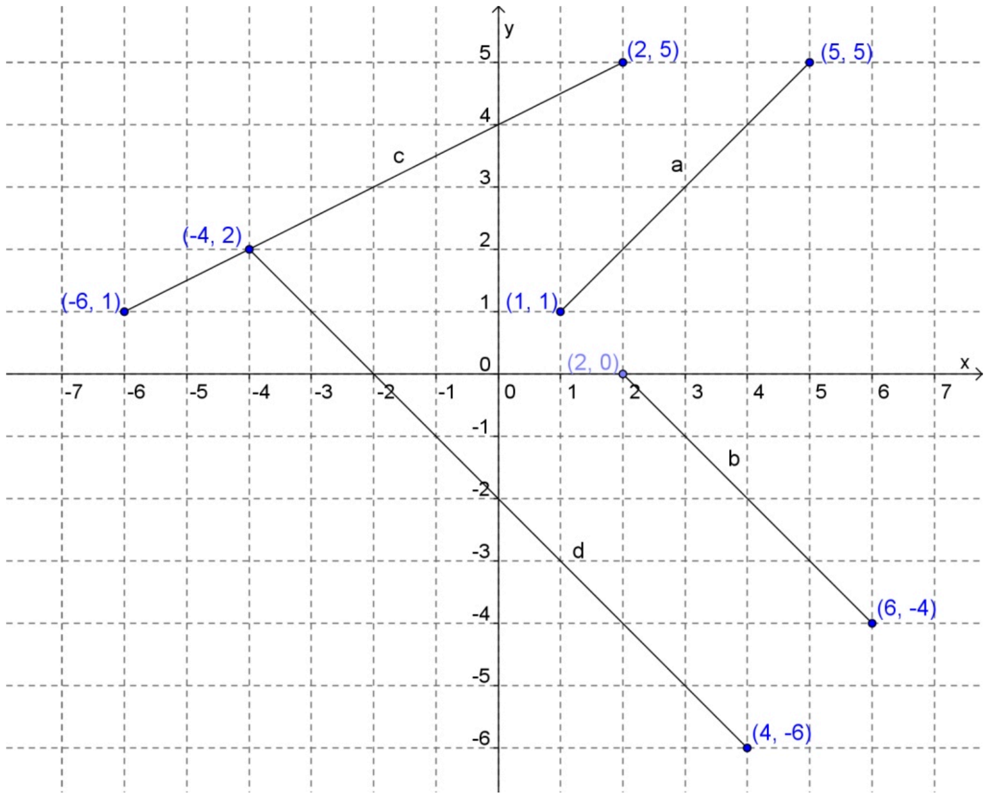


Let's look at a few more lines.

ACTIVITY 2

1. Work out the midpoints of the line segments below, using whatever method you can. (You could use measurement).

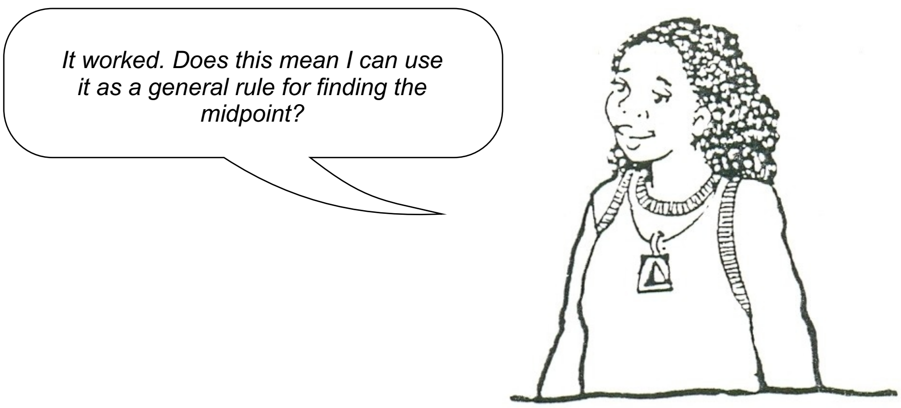
2. Can you see any patterns? If so, write down what you see.



ANSWERS ON PAGE 91

In the first example, if you add the x –coordinates together and divide by 2, you get the x –coordinates of the midpoint.

Is this a general rule? Try looking at the x –coordinates, and the y –coordinates to see if this rule is true of all the lines.



Yes, you can. Therefore the general rule for finding the coordinates of the midpoint of the line segment joining $(x_1;y_1)$ and $(x_2;y_2)$ is

$$x=\frac{(x_1+x_2)}{2};\ y=\frac{(y_1+y_2)}{2}$$

Example
Calculate the coordinates of the midpoint of the line joining the points (–3; 1) and (1; 5).

Solution
There are a few ways of solving this problem. One way is to draw a graph and measure. But this might not be absolutely accurate and may take a lot of time. The other efficient way is to use the algebraic formula.

To solve this we substitute (–3; 1) and (1; 5) for x and y in the formula.

$$x=\frac{(x_1+x_2)}{2} \qquad y=\frac{(y_1+y_2)}{2}$$

$$\therefore x=\frac{-3+1}{2} \quad \text{and} \quad \therefore y=\frac{1+5}{2}$$

$$\therefore x=\frac{-2}{2} \qquad \therefore y=\frac{6}{2}$$

$$\therefore x=-1 \qquad \therefore y=3$$

$\therefore$ the coordinates of the midpoint are (–1; 3).

ACTIVITY 3

Find the coordinates of the midpoint of the line joining these points:

1. (–2; 3) and (6; 3)

2. (4; –1) and (–1; 3)

3. (0; 0) and (3; –8)

ANSWERS ON PAGE 91

The point of intersection of two lines

We can find the point of intersection of two lines by using simultaneous equations or we can put the two lines equal to each other because at that point the y –values are equal too.

Let's take a look at an example.

Example
Find the coordinates of the point of intersection of $y=x+6$ and $y=-3x-2$.

Always write the equation in the form of $y = mx + c$ before you start.

Solution

At the point of intersection the y –values are equal to each other therefore we can say that

$$\begin{aligned} x + 6 &= -3x + 2 \\ x + 3x &= 2 - 6 \\ 4x &= -4 \\ x &= -1 \end{aligned}$$

In order to calculate the y –coordinates of the point of intersection, you will need to substitute the x –coordinates you have just found into one of the given equations.

Take $y = x + 6$ and substitute $x = -1$
$y = (-1) + 6$
$y = 5$

Therefore the point of intersection is (–1; 5).

What would happen if I put $x = -1$ into the other equation?

Well, let's take a look.

Take $y = -3x + 2$ and substitute $x = -1$.

$y = -3(-1) + 2$
$y = 3 + 2$
$y = 5$

The point of intersection is still (–1; 5) so using either equation is correct since they both give the same answer.

ACTIVITY 4

1. Find where the following pairs of lines cut each other.

   a) $y = x + 1$ and $y = -3x + 5$
   b) $y = 2x + 3$ and $y = x - 5$
   c) $y = x - 3$ and $3x + y - 2 = 0$

2. Where does the line $y = 2x - 3$ cut the line through the points (–2; 6) and (3; –2)?

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You have now completed the course on coordinate geometry so perhaps it is useful to highlight some of the more important formulae and their uses. You will recognise some of the information from Unit 4 and some you have learnt in the last three lessons.

FORMULA	USE
$y = mx + c$	standard form of a straight line graph
$m = \frac{y_2 - y_1}{x_2 - x_1}$	gradient of a line when given two points
$y - y_1 = m(x - x_1)$	to find the equation of a line when given the gradient and a point
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	distance between two points
$(x_3; y_3) = (\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2})$	coordinates of the midpoint of the line joining two points

Use this table to check your formulae. Try the following self assessment exercise. This will give you an indication of what you have learnt in these lessons.

Summary

In this lesson you have focused on calculating the distance between two points, the midpoint of a line, finding coordinates and using the following formulae:

$$y = mx + c$$

$$y - y_1 = m(x - x_1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$(x_3; y_3) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

CHECKLIST

Are you able to:

- find the distance between two points
- calculate the midpoint of a line segment
- find the point of intersection of two lines
- use specific formulae

SELF-CHECK EXERCISE

1. Find the gradients of the lines AB and CD, and then say whether the pairs of lines are parallel, perpendicular, or neither parallel nor perpendicular.

   a) $A(1;1), B(-1;0), C(2;-2), D(4;v1)$
   b) $A(-2;4), B(3;1), C(5;-1), D(-2;-8)$
   c) $A(-2;1), B(2;4), C(-3;-1), D(0;3)$

2. If A is the point (−2; 3) and B is the point (−3; 2), find the distance between A and B.

3. a) Find the equation of the line joining $A(-3;7)$ and $B(1;-2)$.
   b) What is the equation of the line parallel to AB passing through $(-1;2)$?
   c) What is the equation of the line perpendicular to AB passing through $(-1;2)$?

4. $A\ (-2;1), B(3;3)$ and $C(6;-3)$ are the vertices of a triangle.

   Determine:
   a) the coordinates of M, the midpoint of AC
   b) the coordinates of N, the midpoint of AB
   c) the coordinates of P, the midpoint of BC.

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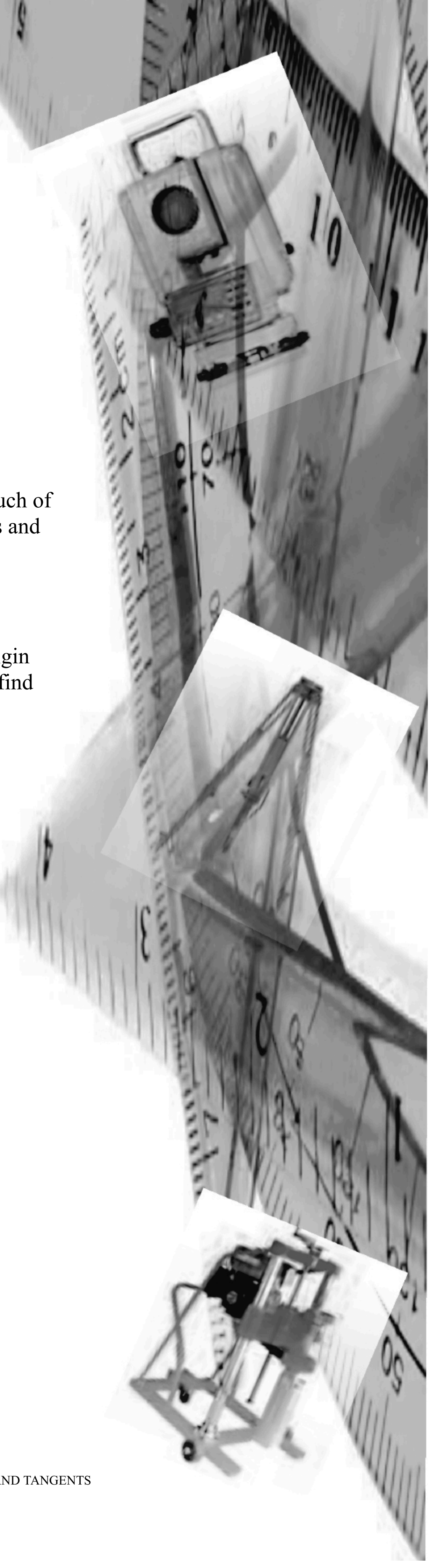
   Plot A, B, C and M, N, P on graph paper. Join ABC.

Well done if you have successfully completed this work. But remember that if you have not fully understood something then it is best to go back now and revise the section again.

LESSON 4

Coordinate Geometry: Circles and Tangents



About this lesson

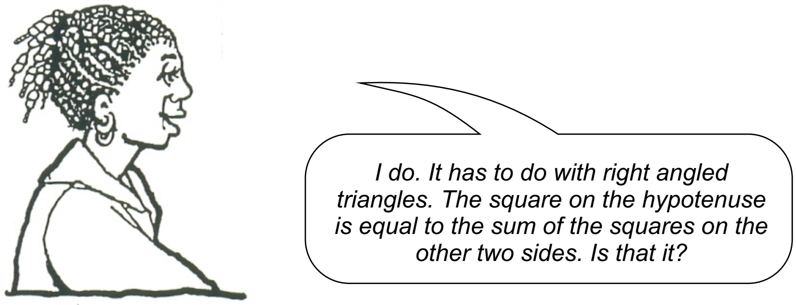
Now that we are comfortable with the principles of coordinate geometry, let's move on to coordinate geometry of the circle. Much of the theory of this section in based on the Theorem of Pythagoras and the Distance Formula.

In this lesson you will:

- determine the equation of a circle that is centred on the origin
- translate the centre of the circle away from the origin and find its equation
- determine the equation of tangents to a circle

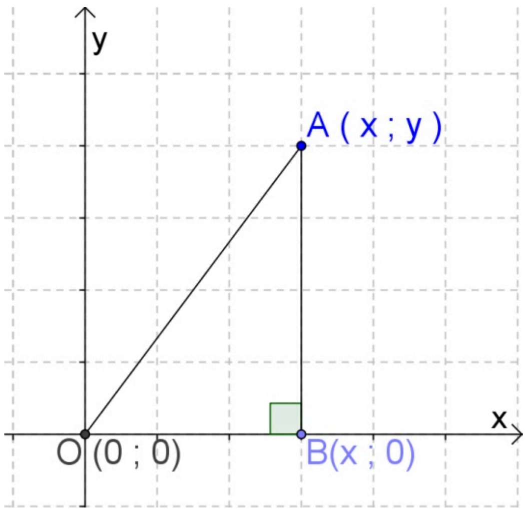
Pythagoras' Theorem

Do you remember the Theorem of Pythagoras?



It is. Well done. We investigated this when finding the distance between two points so this should all be familiar.

The circle



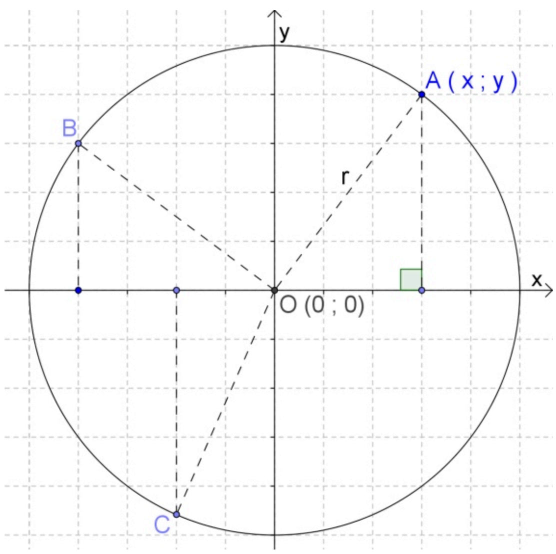
In the diagram alongside, the distance between A and O, also known as the length of AO, can be given by;

$$AO = \sqrt{OB^2 + AB^2}$$
$$= \sqrt{(x-0)^2 + (y-0)^2}$$
$$= \sqrt{x^2 + y^2}$$

The feature of the circle that helps us to determine its equation, is that it is created from a line of constant length, referred to as a radius, that rotates around a fixed point, referred to as the centre. If we start by considering this fixed point to be the origin, the circle would look like this.

The three points along the circle's edge, A, B and C are all equidistant from the origin. If we call the distance from O to A the radius (r), then the line segments OB and OC are also r. Any line segment that we draw from O to the edge of the circle would have the same length, r, and so we can represent the equation of the circle as follows:

$$r^2 = x^2 + y^2$$



Example
Find the equation of a circle centred on the origin that goes through the point P (3; 4).

Solution
Since this is a circle, the length of the line between the origin and this point P, is the radius and remains constant.

We know that the point $P(3; 4)$ has an x –value of 3 and a y –value of 4 so the calculation is as follows;

$$
\begin{aligned}
r^2 &= x^2 + y^2 \\
\therefore r^2 &= (3)^2 + (4)^2 \\
&= 9 + 16 \\
&= 25 \\
\therefore\ r &= 5
\end{aligned}
$$

This means that the radius of the circle is 5 and the equation is given by:
$x^2 + y^2 = 25$

ACTIVITY 1

Find the equation of the following circles centred on the origin.

1. Through the point $Q(5; -12)$
2. Cutting the x –axis at 7
3. Cutting the y –axis at -3
4. With a radius of $2\frac{1}{2}$

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We can also use the equation to find out other things about the circle and we can sketch it.

Example
Sketch the following: $x^2 + y^2 = 34$

Solution
We can tell from the equation that the radius of this circle is $\sqrt{34}$.

As with all sketches, the best way to find the cuts on x and y –axes is to let y and x respectively be equal to zero.

Cut on the x –axis:
Let $y = 0$

$\therefore x^2 + (0)^2 = 34$

$\therefore x^2 = 34$

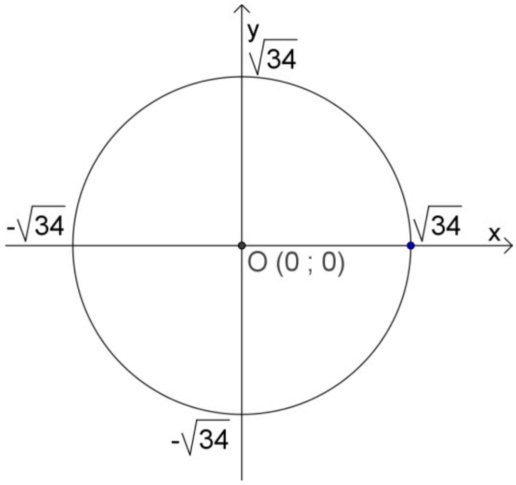
$\therefore x = \pm\sqrt{34}$

Cut on the y –axis:
Let $x = 0$

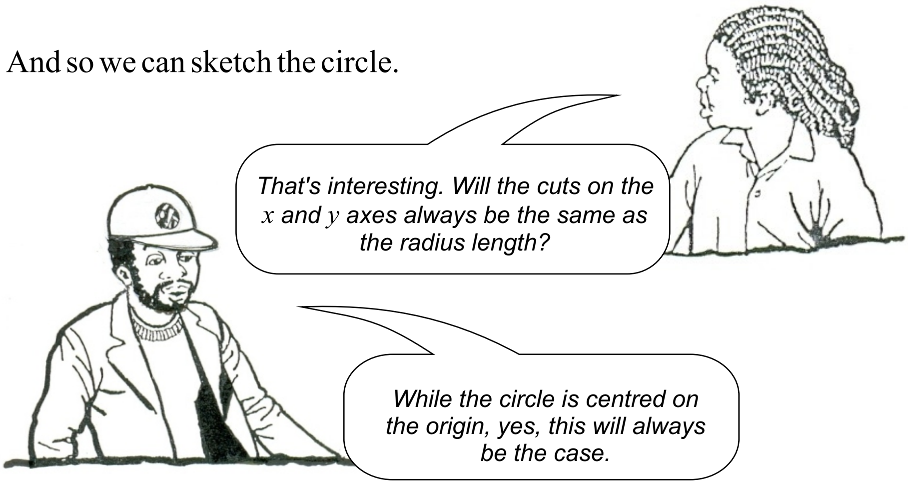
$\therefore (0)^2 + y^2 = 34$

$\therefore y^2 = 34$

$\therefore y = \pm\sqrt{34}$



And so we can sketch the circle.



Example

The point $(-2; p)$ is on the edge of the circle $x^2 + y^2 = 10$. Find the possible value(s) of p.

Solution

When square rooting both sides of an equation, we need to consider both the positive and negative options.

$x = -2$ and $y = p$ $\quad \therefore (-2)^2 + p^2 = 10$

$\therefore 4 + p^2 = 10$

$\therefore p^2 = 6$

$\therefore p = \pm\sqrt{6}$

Try the following activity to check whether you understand everything so far.

ACTIVITY 2

1. Sketch the following circles, showing the intercepts with both axes.
   a) $x^2 + y^2 = 36$
   b) $x^2 = 13 - y^2$
   c) $x^2 + y^2 = p$

2. Determine the radius of the circle with equation $x^2 + y^2 = 32$.

3. The point Q (q; 4) is on the circle $x^2 = 15 - y^2$. Find the value (s) of q.

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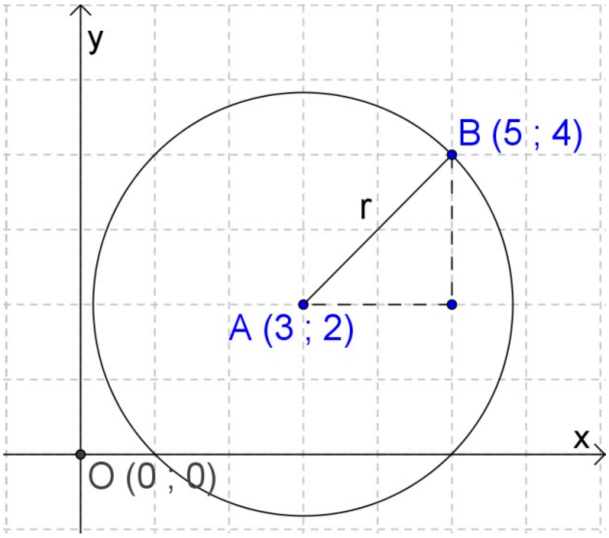
Translating circles

Let us now consider moving the centre away from the origin. The length of the radius still stays constant and the one end still remains fixed. We can therefore use the distance formula to determine the equation. All we need to keep in mind is that the centre is no longer at the origin.



Consider the circle above. We will start by calculating the length of the radius.

$$\begin{aligned} r^2 &= (5-3)^2 + (4-2)^2 \\ &= 4+4 \\ &= 8 \\ \therefore r &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

Now the point B(5; 4) is not the only point on the edge of this circle. There are an infinite number of points so let's generalize the situation and say that the circle passes through the point (x; y). Using this point, the centre of the circle and the length of the radius that we have just found, we can say;

$$(x-3)^2 + (y-2)^2 = \left(2\sqrt{2}\right)^2$$

This equation now represents all the possible coordinates that are $2\sqrt{2}$ units away from the centre (3; 2).

We can simplify this equation to read $(x-3)^2 + (y-2)^2 = 8$.

Is there an equation that will work with a circle that has **any** origin?

Let's see. If we take the centre of the circle to be (a; b), a point on the circle to be (x; y) and the radius to be r, what do we find?

Using the distance formula we get:

$$(x-a)^2+(y-b)^2=r^2$$

And that gives us the general equation for any circle with centre (a; b) and radius r.

Example

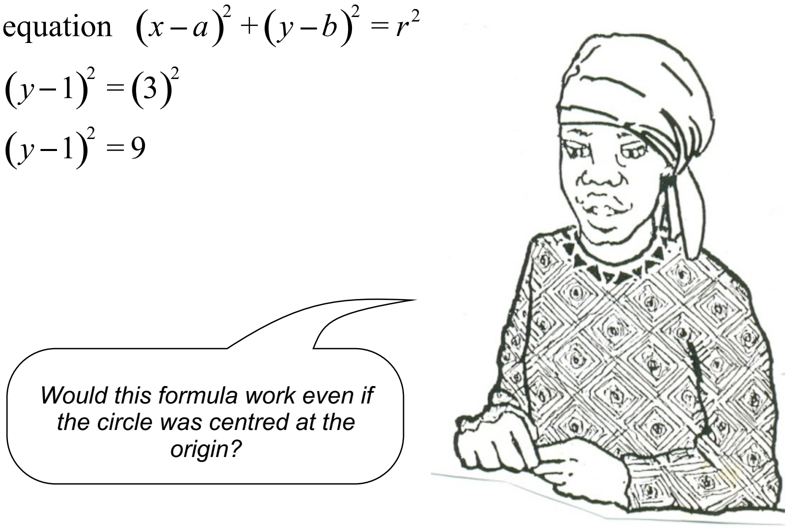
Find the equation of a circle with centre (4; 1) and a radius of 3.

Solution

Using the equation $(x-a)^2+(y-b)^2=r^2$

$(x-4)^2+(y-1)^2=(3)^2$

$(x-4)^2+(y-1)^2=9$



If the centre was at the origin, then the coordinates of the centre would be (0; 0) and the equation would become $(x-0)^2+(y-0)^2=(3)^2$ which simplifies to $x^2+y^2=9$. Since we know this is correct, we can see that this formula does work with any circle, no matter where the centre is.

Example

Determine the equation of the circle with a centre of (2; −1) and going through the point (1; 5).

Solution

Substituting into the general equation we get:

$$(x-2)^2+(y-(-1))^2=r^2$$

$$\therefore\ (x-2)^2+(y+1)^2=r^2$$

The equation is still not complete as we need to find a value for r. To do this, all we have to do is substitute the point that we were given.

$$\therefore\ (1-2)^2+(5+1)^2=r^2$$

$$\therefore\ 1+36=r^2$$

And so the equation of the circle is $(x-2)^2+(y+1)^2=37$.

ACTIVITY 3

1. Write down the centre of each of the following circles and give the length of the radius.

   a) $(x+2)^2 + (y-4)^2 = 16$

   b) $x^2 + (y-3)^2 = 15$

   c) $(x-p)^2 + (y+4)^2 = q$

2. Determine the equation of the circle that passes through the point (–2; 3) with a centre of (3; –4).

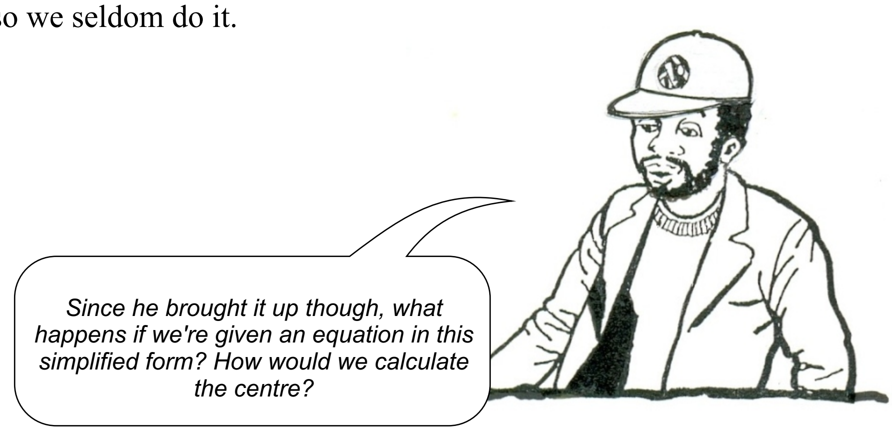
3. Find the equation of a circle with centre (2; –2) that passes through the origin.

ANSWERS ON PAGE 94

We can take an equation like $(x-2)^2 + (y+2)^2 = 8$ and multiply out as follows:

$$x^2 - 4x + 4 + y^2 + 4y + 4 = 8$$
$$x^2 - 4x + y^2 + 4y = 0$$

This equation still represents the same circle as before only now we cannot immediately identify the centre of the circle. So although it is not incorrect to multiply out and collect like terms, it's not helpful and so we seldom do it.



That's a good question, let's take a look.

Consider the equation $x^2 + 2x + y^2 - 4y = 49$. This represents a circle whose centre is not in the origin. How would we find out where the centre is?

To answer this question we need to think about what we need the left hand side of this equation to look like.

We know that if we can get it into the form $(x-a)^2 + (y-b)^2 = r^2$, then we will have the centre and radius available. We need to have a sum of two perfect squares on the left, one involving $x^2 + 2x$ and the other involving $y^2 - 4y$.

The problem we have is that neither of these two expressions are perfect squares so we need to make some changes to our original equation.

To make a perfect square out of $x^2 + 2x$ we need 1 since $x^2 + 2x + 1$ is a perfect square that can be written as $(x+1)^2$. And to make a perfect square from $y^2 - 4y$ we need 4 since $y^2 - 4y + 4$ is a perfect square that can be written as $(y-2)^2$.

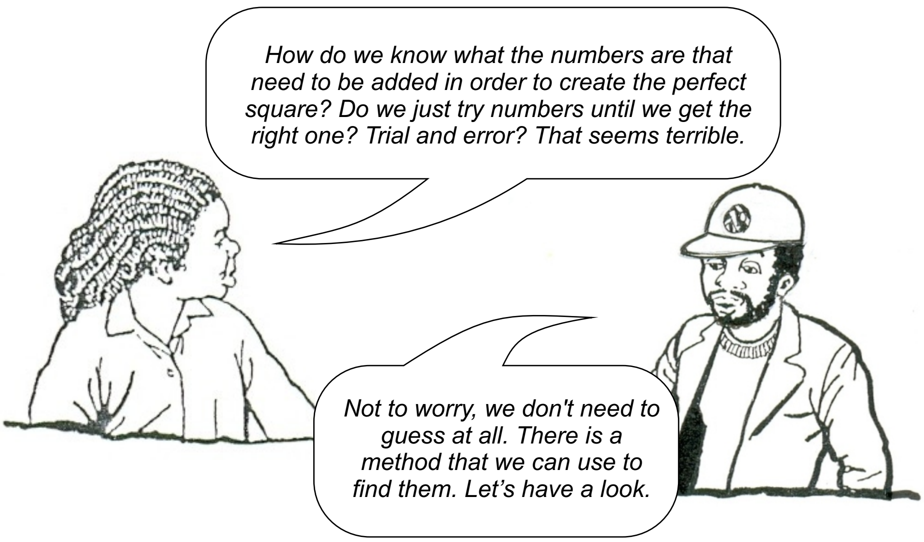
So looking at our original equation $x^2 + 2x + y^2 - 4y = 49$, we see that we must add both 1 and 4 to the left hand side to create perfect squares, and so we need to add both 1 and 4 to the right hand side in order to keep our equation balanced.

Now we have $x^2 + 2x + 1 + y^2 - 4y + 4 = 49 + 1 + 4$

The left hand side has two trinomials which can be factorized and we simplify the right hand side,

$\therefore (x+1)^2 + (y-2)^2 = 54$

which means that the centre of the circle is $(-1;2)$ and the radius is $\sqrt{54}$.



The number needed to make $x^2 + 2x$ a perfect square, will be half of the coefficient of the x term squared. That is $(\frac{1}{2} \cdot 2)^2$. This will always complete the square.

$(\frac{1}{2} \times 2)^2 = 1$, which means that $x^2 + 2x + 1$ is now a perfect square. Of course, if I have 1 to one side of the equation I must be sure to add 1 to the other side of the equation too.

The number needed to make $y^2 + 4y$ a perfect square, will be half of the coefficient of the x term squared. That is $(\frac{1}{2} \times 4)^2$.
$(\frac{1}{2} \times 4)^2 = 4$, which means that $y^2 - 4y + 4$ is now a perfect square.
Adding 4 to other side of the equation will keep the equation balanced.

The equation now looks different but still represents the same circle :
$x^2 + 2x + 1 + y^2 - 4y + 4 = 49 + 1 + 4$
Factorizing the two trinomials on the left hand side gives us
$(x+1)^2 + (y-2)^2$ and simplifying the right hand side gives us 54. And now we have the equation of the circle in standard form.

$$(x+1)^2 + (y-2)^2 = 54$$

From here we simply read off the values for the centre of the circle and square root the right hand side to get the radius.

Let's try that with another equation.

Example
Find the centre of the circle represented by $x^2 - 6x + y^2 + 2y = 15$ as well as the length of the radius.

Solution
$x^2 - 6x + y^2 + 2y = 15$ needs to be written in the standard form so that we can see the centre. This means I need to create 2 perfect squares on the left hand side.

$x^2 - 6x$ needs to have $(\frac{1}{2} \times 6)^2$ to complete the square which is 9.
$y^2 + 2y$ needs to have $(\frac{1}{2} \times 2)^2$ to complete the square which is 1.

So we need to add 9 and 1 to the left hand side to complete both squares and therefore we need to add them both to the right hand side as well.

$$x^2 - 6x + 9 + y^2 + 2y + 1 = 15 + 9 + 1$$

Then we factorize both trinomials on the left hand side and simplify the right hand side.

$$(x-3)^2 + (y+1)^2 = 25$$

And so we can deduce that the centre of the circle is at (3; −1) and the radius is 5.

Here is another example.

Example

Find the coordinates of the centre of circle $x^2 + x + y^2 - 8y = 8$ as well as the radius.

Solution

$$x^2 + x + (\frac{1}{2} \times 1)^2 + y^2 - 8y + (\frac{1}{2} \times 8)^2 = 8 + (\frac{1}{2})^2 + (4)^2$$

Complete the square on both sides

$$\therefore\ x^2 + x + \frac{1}{4} + y^2 - 8y + 16 = 8 + \frac{1}{4} + 16$$

Simplify both sides

$$\therefore\ (x + \frac{1}{2})^2 + (y - 4)^2 = \frac{97}{4}$$

Factorize the trinomials on the left

$\therefore$ centre is $(-\frac{1}{2}; 4)$ and the radius is $\sqrt{\frac{97}{2}}$

Now try Activity 4.

ACTIVITY 4

Determine the centre of each of the circles below and the length of each radius.

1. $x^2 + 2x + y^2 - 6y = 6$
2. $x^2 + 4x + y^2 + 2y - \frac{2}{3} = 0$
3. $2x^2 + 2y^2 + 4y - 6x - 17 = 0$

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We are now going to work with tangents.

Tangents

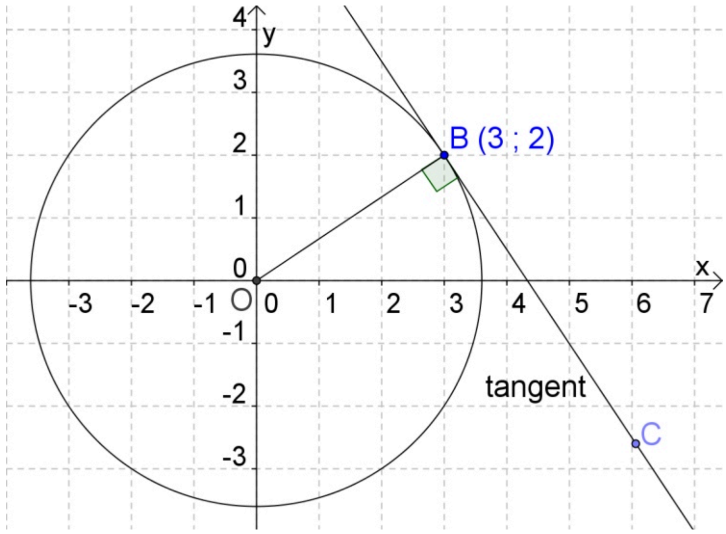
A tangent is a line that intersects a curve, but only cuts the curve at one point, and that's where the concept of 'touches' comes from. It doesn't cut through the curve, it merely touches the edge of the curve. In this lesson, we're only interested in tangents that touch circles.

Finding the equation of a tangent to a circle

We refer to the point where the straight line touches the circle as the 'point of contact'.

In the diagram on the next page, point *B* would be the point of contact. At this point, the straight line touches the edge of the circle at just one point, in this instance at the point (3; 2), making the straight line *BC* a tangent.

The very useful fact about tangents that we use, is that the angle at which the radius of the circle hits the tangent at the point of contact is 90°. Below this is shown as $O\hat{B}C = 90°$



Let's see if we can find the equation of the circle above. It is centred on the origin and passes through the point (3; 2) which means that $x = 3$ and $y = 2$.

$$\therefore (3)^2 + (2)^2 = r^2$$

$$\therefore r^2 = 13$$

The equation of the circle is therefore $x^2 + y^2 = 13$.

Now we want to find the equation of the tangent to the circle at the point (3; 2).
Since a tangent is a straight line, the equation must take on the form $y = mx + c$. In this equation we need to find values for both m and c where m is the gradient of the line and c is the y –intercept.

Let's first look at the problem of finding the gradient of the tangent. What we do know is that the tangent is at 90° to the radius drawn to the point of contact. This helps us since if we can find the gradient of the radius, then we can calculate the gradient of the tangent using the fact that the product of perpendicular gradients is -1.

So we start with the gradient of the radius.

$$m_{OB} = \frac{2-0}{3-0}$$

$$= \frac{2}{3}$$

If the gradient of the radius is $\frac{2}{3}$, then the gradient of the tangent must be $-\frac{3}{2}$.

Let's check if that is correct. $m_{OB} \text{ x } m_{BC} = \frac{2}{3} \text{x} - \frac{3}{2}$

$= -1$ which is correct.

The equation of the tangent can now be written as $y = -\frac{3}{2}x + c$. All we need to do to find is substitute a point on the line into the equation. We have the point of contact, which is on both the tangent and the circle. Let's use that point. As we said before, the point (3; 2) means that $x = 3$ and $y = 2$.

$$2 = -\frac{3}{2}(3) + c$$

$$\therefore\ 2 + \frac{9}{2} = c$$

$$\therefore\ c = \frac{13}{2}$$

And the equation of the tangent is $\therefore y = -\frac{3}{2}x + \frac{13}{2}$.

or $2y = -3x + 13$

It probably looks rather complicated, but it isn't. Let's look at another one.

Example

Find the equation of the tangent to the circle $x^2 + y^2 = 25$through the point (3;–4).

Solution

First find the gradient of the radius to the point of contact is $m_r = \frac{0-(-4)}{0-3} = -\frac{4}{3}$

Use this gradient to determine the gradient of the tangent $m_t = \frac{3}{4}$

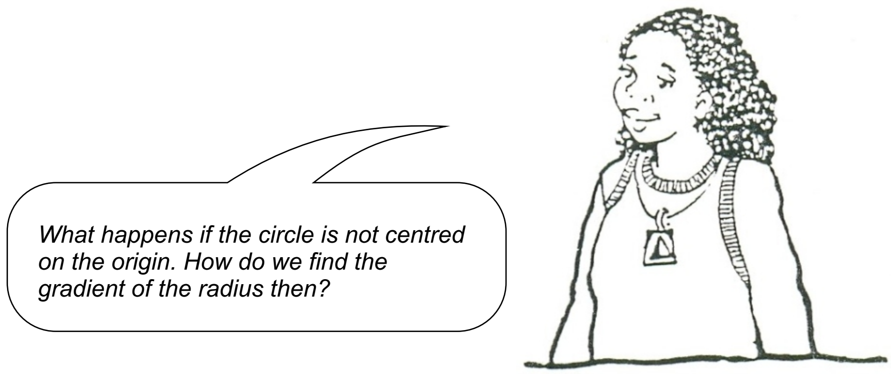
The equation of the tangent is now $y = \frac{3}{4}x + c$

Use the point of contact to substitute for x and y $-4 = \frac{3}{4}(3) + c$

$$\therefore -4 - \frac{9}{4} = c$$

$$\therefore c = -\frac{25}{4}$$

The equation of the tangent is $y = \frac{3}{4}x - \frac{25}{4}$ or $4y = 3x - 25$



No problem, we just have to work a little harder to get there. We know how to find the centre of any circle so it shouldn't cause us too much trouble. Let's try.

Example
Find the equation of a tangent to the circle $(x-5)^2+(y-2)^2=17$ at the point (1; 1).

Solution
The centre of the circle is (5; 2), so the gradient of the radius to the point of contact is $m_r=\frac{2-1}{5-1}=\frac{1}{4}$

Therefore, the gradient of the tangent is –4.

Using the point of contact (1; 1), we get $1=-4(1)+c$

$$c=5$$

The equation of the tangent is $y=-4x+5$

ACTIVITY 5

1. Find the equation of the tangent to the circle $(x-3)^2+(y+2)^2=5$ through the point (1; –3).

2. Find the equation of the tangent to the circle $x^2+4x+y^2+2y=5$ through the point (1; –2).

3. Find the equation of the tangent to the circle $2x^2+8x+2y^2-4y=10$ through the point (–1; 4).

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Summary

This lesson focuses on sketching a circle, finding the centre of a circle, the length of a radius of a circle and working with tangents.

If the self-check exercise gives you any trouble, go back to the explanations and activities to get better clarity.

CHECKLIST

Are you able to:

- determine the equation of a circle that is centred on the origin
- translate the centre of the circle away from the origin and find its equation
- determine the equation of tangents to a circle

SELF-CHECK EXERCISE

1 radius
2 radii

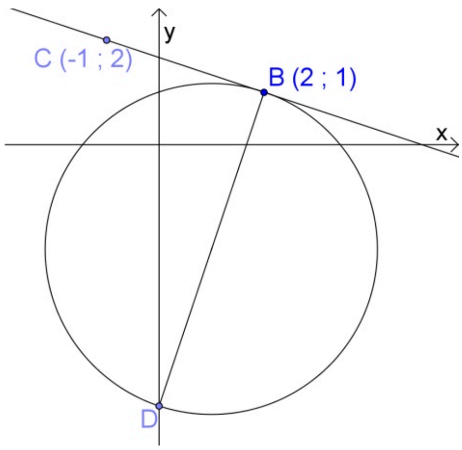
1. Determine the centre of each of the following circles as well as the length of their radii.

   a) $x^2 + y^2 = 17$
   b) $(x-2)^2 + (y-3)^2 = 9$
   c) $x^2 + 4x + y^2 + 4y - 18 = 0$
   d) $x^2 + y^2 + 2y - 6 = 0$

2. Find the equation of the circle with a diameter AB where A is the point $(-1; 1)$ and B is the point $(5; -3)$. (Hint: First find the centre of the circle.)

3. Find the equation of the tangent drawn to the circle $x^2 - 4x + y^2 = 0$ through the point (2; 2).

4. Use the following sketch to:



   a) Determine the equation of straight line BC which is a tangent to the circle at B.
   b) If BD is the diameter of the circle, find the equation of BD.
   c) Determine the equation of the circle.

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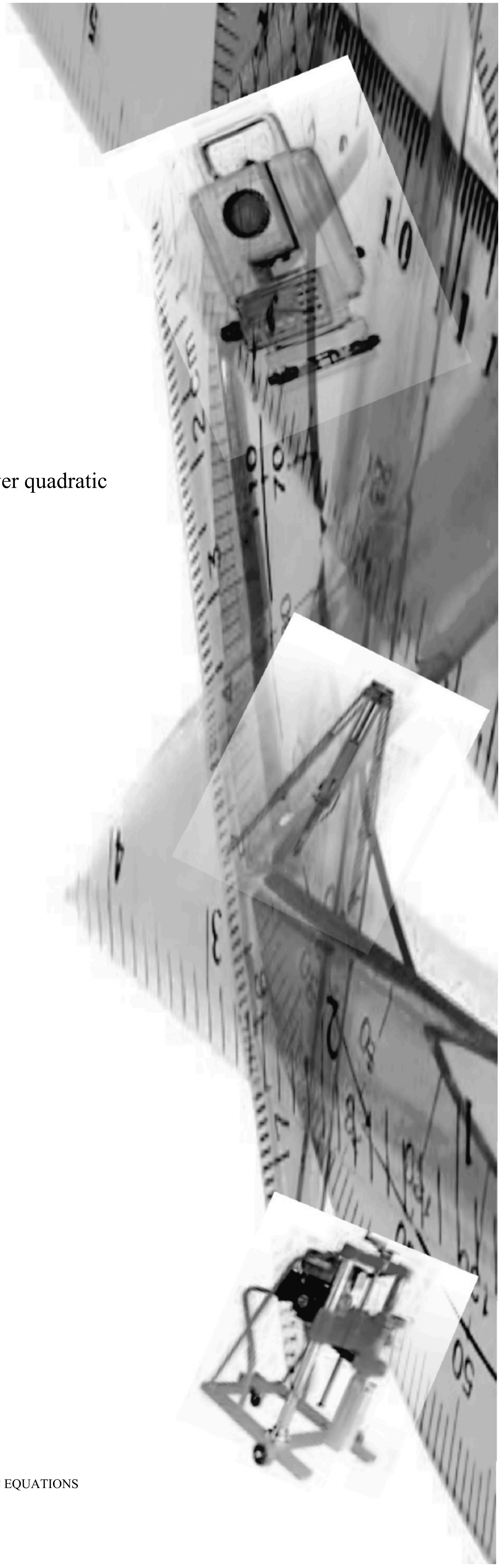
LESSON 5

Quadratic Equations



About this lesson

Welcome to Lesson 5. In this lesson we are going to cover quadratic equations.

In this lesson, you will:

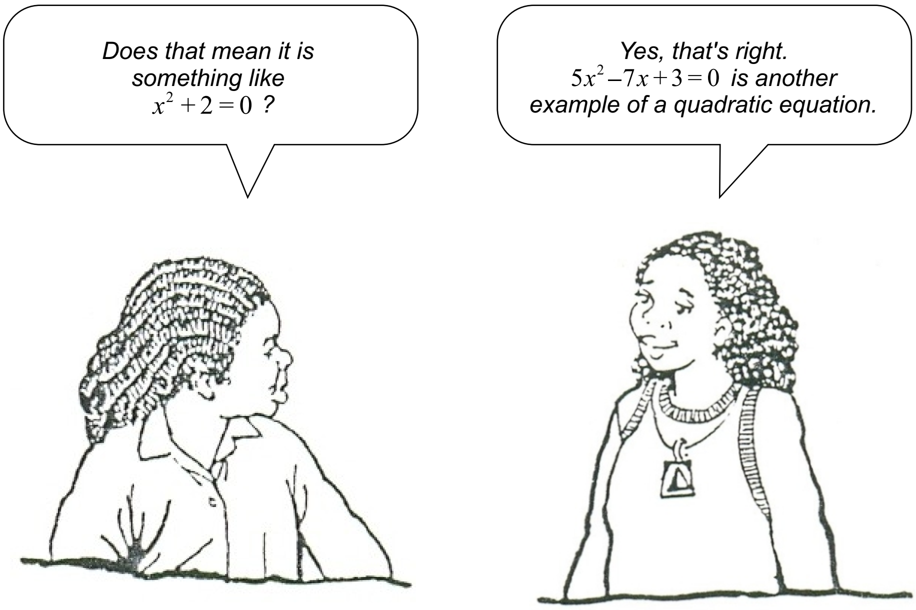
- solve quadratic equations by factorising
- solve quadratic equations by completing the square
- solve quadratic equations by using a formula
- solve quadratic equations with fractions

What is a quadratic equation?

> *A quadratic equation is an equation that has a standard form of $ax^2+bx+c=0$ where a, b and c are real numbers.*
>
> *The number a cannot be equal to 0.*

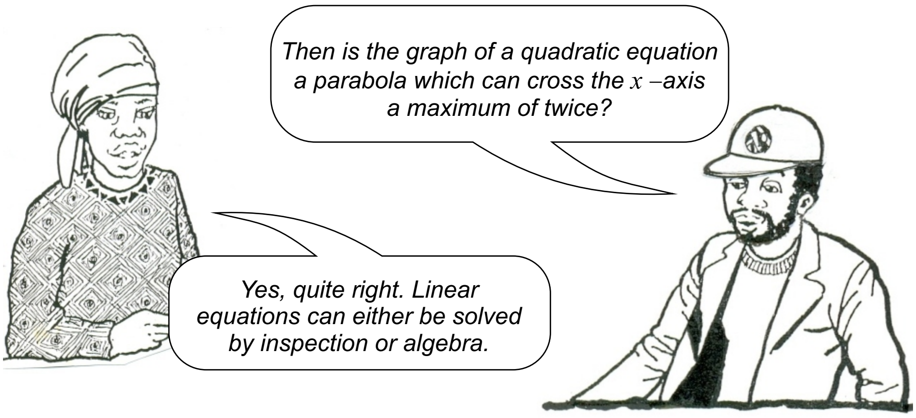
A quadratic equation is a second degree equation; it is an equation in which the highest power of the unknown is 2.

x^2+2 is a quadratic expression but $x^2+2=0$ is a quadratic equation.



You will understand quadratic equations if you compare them with linear equations. $2x+6=0$ is a linear equation; it has only one solution and none of the variables (x) are raised to a higher degree than one.

This means that the linear equation will be a straight line, which only crosses the x –axis once. If we look at the example of a quadratic equation, $5x^2-7x+3=0$, the highest power is 2. So, quadratic equations have a maximum of two solutions or roots.



However there are many ways of solving quadratics. In this lesson we will look at some of the different methods.

Solving quadratic equations by factors

You have already learnt how to factorise the expression $ax^2 + bx + c$, in a number of situations. Look at the following examples:

Example 1

$b = 0,\ a,\ c \neq 0$

Solve by factors $x^2 = 16$

Solution

$x^2 = 16$

$\therefore\ x^2 - 16 = 0$ (take all terms to the LHS with the RHS = 0)

$(x+4)(x-4) = 0$ (factorise using the difference of two squares)

$\therefore\ x - 4 = 0 \text{ or } x + 4 = 0$

$\therefore\ x = 4 \text{ or } x = -4$

Check your solution by substituting the values of x into the equation and see if LHS = RHS

A basic principle for solving quadratic equations is, if A x $B = 0$ then $A = 0$ or $B = 0$.

Example 2

$c = 0, a, b \neq 0$

Solve by factors $3x^2 - 12x = 0$

Solution

$3x^2 - 12x = 0$ (standard form)

$3x(x-4) = 0$ (common factor of $3x$)

$\therefore 3x = 0 \text{ or } x - 4 = 0$ (basic principle)

$\therefore\ x = 0\ \ x = 4$ (check your answers)

Example 3

$a = 1,\ b,\ c \neq 0$

Solve by factors $x^2 + 2x - 3 = 0$

Solution

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$\therefore\ x + 3 = 0 \text{ or } x - 1 = 0$

$\therefore\ x = -3 \text{ or } x = 1$ (check your answers)

You cannot put $x^2 + 2x = -3$ and solve for x. For the basic principle to work, the equation must be equal to zero.

Example 4

$a > 1,\ b,\ c \neq 0$

Solve by factors $6x^2 + 13x = 5$

Solution

Method: Take everything to the LHS so that RHS = 0
Then factorise the LHS.

LHS = left hand side
RHS = right hand side

$6x^2 + 13x = 5$

$\therefore\ 6x^2 + 13x - 5 = 0$

$\therefore\ (3x - 1)(2x + 5) = 0$

$\therefore\ 3x - 1 = 0 \ or \ 2x + 5 = 0$

$\therefore\ x = \frac{1}{3} \quad or \quad x = -\frac{5}{2}$

To be able to solve quadratic equations easily and quickly using factors you must know how to factorise trinomials.

It is important to remember that *a* can also be less than 1, but when solving, it is important to multiply both the LHS and RHS by −1 before you start. Here is an example.

Example 5

$a < 1,\ b,\ c \neq 0$

Solve by factors $-2x^2 + 11x = 5$

Solution

$-2x^2 + 11x - 5 = 0$	(take all terms to LHS, make RHS = 0)
$\therefore\ 2x^2 - 11x + 5 = 0$	(multiply both sides by -1)
$\therefore\ (2x - 1)(x - 5) = 0$	(factorise as usual)
$\therefore\ 2x - 1 = 0 \ or \ x - 5 = 0$	
$\therefore\ x = \frac{1}{2} \qquad x = 5$	(check your answers)

Steps involved in factorising quadratic equations

1. Make sure that all the terms are taken to the LHS and that the RHS = 0.

2. If a < 1, multiply the equation by −1 before factorising.

3. Take out the common factor if possible.

4. Factorise.

5. **Always** check the solution to make sure you have factorised correctly.

6. All the answers to the above questions are referred to as roots of quadratic equations or roots.

ACTIVITY 1

Solve the following by factorisation:

Not all variables have to be x.

1. $3a^2 + 12a = -9$

2. $4m^2 - 25 = 0$

3. $p^2 = 4p$

4. $2(9x - 4) = 9x^2$

5. $x^2 + 4x - 3 = 0$

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We have to find other methods of solving quadratic equations that can't be solved by factors. Here is a method where you find the square root of both sides. Let's look at the following cases:

1. $x^2 = 81$

   $\therefore\ x = \pm 9$ (square root on both sides)

   $\therefore\ x = 9 \quad or \quad x = -9$

2. $(x-5)^2 = 16$

   $\therefore\ x - 5 = \pm 4$ (square root on both sides)

   $\therefore\ x - 5 = 4 \quad or \quad x - 5 = -4$

   $\therefore\ x = 9 \quad or \quad x = 1$

3. $x^2 + 14x + 49 = 25$

   $\therefore\ (x+7)^2 = 25$

   $\therefore\ x + 7 = \pm 5$ (square root on both sides)

   $\therefore\ x + 7 = 5 \quad or \quad x + 7 = -5$

   $\therefore\ x = -2 \quad or \quad x = -12$

The perfect square

What is a perfect square? If you look at the examples above you will see that there is a perfect square on the LHS and on the RHS. You were then able to take the square root on each side.

Well, 9 + 16 are perfect squares. $3 \times 3 = 9$ and $4 \times 4 = 16$. Also a^2 and x^2 are perfect squares since $a \times a = a^2$ and $x \times x = x^2$.

If you look at the examples above you will see that there is a perfect square on the LHS and on the RHS. Also, the square root on both sides was taken.

Let's look at the following equation $x^2 - 4x - 21 = 0$.

Although this equation can be solved by factorisation, we can also solve it by 'taking the square roots' on both sides provided we can write the equation another way.

This process of transforming the equation is known as 'completing the square'.

Solving quadratic equations by completing the square

Example 6

Solve $x^2 - 4x - 21 = 0$ by completing the square

Complete the square by adding to both sides the square of half the co-efficient of x.

Solution

$x^2 - 4x - 21 = 0$

$\therefore\ x^2 - 4x = 21$ (take variable terms to LHS, constant to RHS)

$\therefore\ x^2 - 4x + (2)^2 = 21 + (2)^2$ (complete the square)

$\therefore\ x^2 - 4x + 4 = 25$

$\therefore\ (x - 2)(x - 2) = 25$

$\therefore\ (x - 2)^2 = 25$ (simplify the LHS and the RHS)

$\therefore\ x - 2 = \pm 5$ (take the square root on both sides)

$\therefore\ x - 2 = 5 \quad or \quad x - 2 = -5$

$\therefore\ x = 7 \quad or \quad x = -3$

Example 7

Solve by completing the square: $2x^2 - 7x + 4 = 0$

Solution

$2x^2 - 7x + 4 = 0$

$\therefore\ x^2 - \frac{7}{2}x + 2 = 0$ (always make the coefficient of $x^2 = 1$ when completing the square)

$\therefore\ x^2 - \frac{7}{2}x = -2$ (take terms in x to LHS and the constant to RHS)

$\therefore\ x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 = -3 + \left(\frac{7}{4}\right)^2$ (complete the square by adding to both sides the square of half the coefficient of x)

$\therefore\ \left(x - \frac{7}{4}\right)\left(x - \frac{7}{4}\right) = -3 + \frac{49}{16}$

$\therefore\ \left(x - \frac{7}{4}\right)^2 = \frac{1}{16}$ (simplify the LHS and RHS)

$\therefore\ x - \frac{7}{4} = \pm\frac{1}{4}$ (take the square root on both sides)

$\therefore\ x - \frac{7}{4} = \frac{1}{4}$ or $x - \frac{7}{4} = -\frac{1}{4}$

$\therefore\ x = \frac{1}{4} + \frac{7}{4}$ or $x = \frac{1}{4} - \frac{7}{4}$

$\therefore\ x = 2$ or $x = \frac{3}{2}$

These roots are irrational.

ACTIVITY 2

Solve by completing the square $x^2 + x - 11 = 0$.

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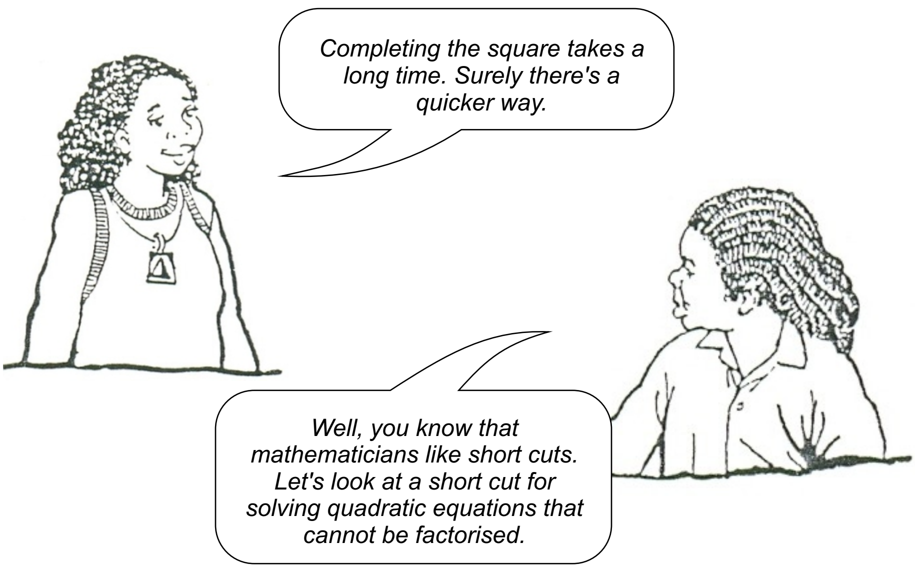


Solution of quadratic equations using the quadratic formula

So far we have been solved equations through factorisation or completing the square. What happens when you cannot factorise and you need a faster method than completing the square? There is a formula that we can use to find the roots of a quadratic equation. This formula is known as the quadratic formula.

If you are given a quadratic equation $ax^2 + bx + c = 0$, the roots of this equation can be found using the formula:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The formula can be shortened this way:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's look at the following examples:

Example 8
Calculate the roots of the following equations using the formula:

$x^2 - 6x - 7 = 0$

Solution
Let us look at the given equation $x^2 - 6x - 7 = 0$. We can see that $a = 1$, $b = -6$ and $c = -7$.

Using the formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-7)}}{2(1)} \\ &= \frac{6 \pm \sqrt{36 + 28}}{2} \\ &= \frac{6 \pm \sqrt{64}}{2} \end{aligned}$$

$$\therefore \; x = \frac{6+8}{2} \quad or \quad x = \frac{6-8}{2}$$

$$\therefore \; x = 7 \quad or \quad x = -1$$

Is this answer correct? Let us check. Did you notice that you can factorise the quadratic expression given here?

$x^2 - 6x - 7 = 0$

$\therefore\ (x-7)(x+1) = 0$

$\therefore\ x - 7 = 0 \quad or \quad x + 1 = 0$

$\therefore\ x = 7 \quad or \quad x = -1$

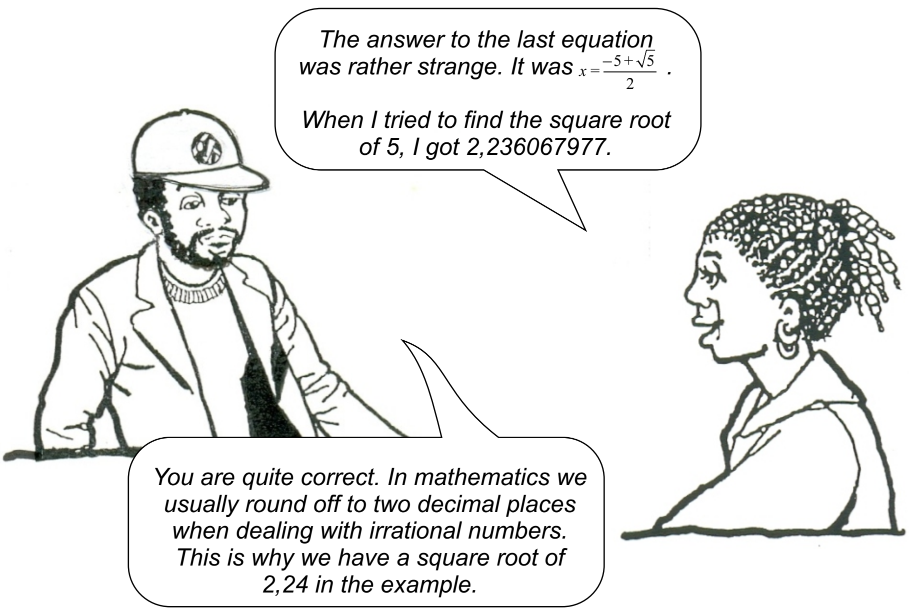
So the formula gives correct answers. Let us look at another example.

Example 9
Solve for x: $x^2 + 5x + 5 = 0$

Solution
$a = 1, b = 5$ and $c = 5$

$$x_{1,2} = \frac{-(5) \pm \sqrt{(5)^2 - 4(1)(5)}}{2(1)}$$
$$= \frac{-5 \pm \sqrt{25 - 20}}{2}$$
$$= \frac{-5 \pm \sqrt{5}}{2}$$
$$\therefore x_1 = \frac{-5+\sqrt{5}}{2} \quad or \quad x_2 = \frac{-5-\sqrt{5}}{2}$$
$$\therefore x_1 = -1,38 \quad or \quad x_2 = -3,62 \qquad \text{(correct to 2 decimal places)}$$

Irrational numbers are numbers that cannot be written as a ratio of two integers. Examples are $\sqrt{5}$ and $\sqrt{3}$.



You have had two examples of how the formula works. How about an activity?

ACTIVITY 3

Where do the following functions cut the x - axis?

1. $y = x^2 - 7x + 12$

2. $y = 3x^2 - 8$

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3. $y = 2x^2 + 6x + 3$

Remember, functions cut the x –axis at points where $y = 0$. This means that the question is looking for values of the x –coordinates where $y = f(x) = 0$. These questions are therefore asking for the solution of the equation $f(x) = 0$.

It is usually sufficient to give only the x –coordinate when you are giving a point where the graph cuts the x –axis. This is so because all the y –coordinates at these points are zero. For example, the answer to no 1 can be written as (4; 0) and (3; 0).

Solving quadratic equations with fractions

If an equation has fractions, get rid of the fractions by multiplying throughout by the LCM. This is the lowest common multiple, a term into which all denominators can be divided.

Let's look at a couple of examples.

Example 10

Solve for a in $6a - 7 = \frac{5}{a}$

Solution

$6a - 7 = \frac{5}{a}$

$6a^2 - 7a = 5$ (multiply both sides by LCM which is)

$6a^2 - 7a - 5 = 0$ (all terms on LHS and RHS = 0)

$\therefore\ (3a - 5)(2a + 1) = 0$

$\therefore\ 3a - 5 = 0 \quad or \quad 2a + 1 = 0$

$\therefore\ a = \frac{5}{3} \quad or \quad a = -\frac{1}{2}$

Example 11

Solve for x: $\dfrac{5(x-1)}{x^2+x-6}-\dfrac{4}{4-x^2}=\dfrac{4}{x+3}$

> *Do not make the mistake of changing $\dfrac{-4}{4-x^2}$ to $\dfrac{-4}{x^2-4}$*
> *If you take out –1 at the bottom, $\dfrac{-4}{4-x^2}$ changes to $\dfrac{4}{x^2-4}$*

Solution

$\dfrac{5(x-1)}{x^2+x-6}-\dfrac{4}{4-x^2}=\dfrac{4}{x+3}$

$\therefore \dfrac{5(x-1)}{x^2+x-6}+\dfrac{4}{x^2-4}=\dfrac{4}{x+3}$ (Switch around LHS)

$\therefore \dfrac{5(x-1)}{(x+3(x-2)}+\dfrac{4}{(x-2)(x+2)}=\dfrac{4}{x+3}$ (factorise the denominators)

$\therefore 5(x-1)(x+2)+4(x+3)=4(x-2)(x+2)$ (multiply by LCM which is $(x + 3)(x-2)(x + 2)$)

$\therefore 5(x^2+x-2)+4x+12=4(x^2-4)$

$\therefore 5x^2+5x-10+4x+12-4x^2+16=0$

$\therefore x^2+9x+18=0$

$\therefore (x+6)(x+3)=0$

$\therefore x+6=0 \quad or \quad x+3=0$

$\therefore x=-6 \quad or \quad x=-3$

If you put $x = -3$ into the original equation, you get a denominator of 0 on the LHS so the equation is undefined, leaving $x = -6$ as the only solution.

ACTIVITY 4

> *Don't forget to check the roots.*

Solve the following:

1. $\dfrac{x}{x-3}=\dfrac{6}{x-5}$

2. $\dfrac{4}{3x-2}-\dfrac{3}{2x-3}=\dfrac{1}{2x-1}$

3. $\dfrac{x}{3x-6}-\dfrac{2}{2-x}=2x$

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Summary

In this lesson you learnt:

- to write the equation in standard form
- if $a = -1$ and you are factorising then you need to multiply the equation by -1
- to take out any common factors first
- when completing the square to make sure that the coefficient of x^2 is 1 before you start

- that the formula for solving a quadratic equation is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- if the equation has fractions to get rid of the fractions by multiplying by the LCM of the denominators
- to solve by factors if factors can be found; otherwise to use the formula
- to only solve by completing the square if you are asked to do so
- to solve quadratic equations by factorisation, completing the square, and using the quadratic formula
- simplify and solve a quadratic equation that has fractions.

CHECKLIST

Are you able to:

- solve quadratic equations only by completing the square
- solve quadratic equations by factorisation, completing the square and using the quadratic formula
- simplify and solve a quadratic equation that has factions.

SELF-CHECK EXERCISE

1. Is the solution to the equation $x^2 - 5x - 6 = 0$

   a) $x = 6;1$
   or b) $x = 6;-1$
   or c) $x = -3;-2$

2. Solve for x

   a) $x^2 - 4x + 4 = 0$ by factorising

   b) $x^2 + 14x = 32$ by completing the square

   c) $3x^2 - 4x - 2 = 0$ by formula

   d) $\frac{x+2}{x-2} = 1 - \frac{x+8}{x+2}$

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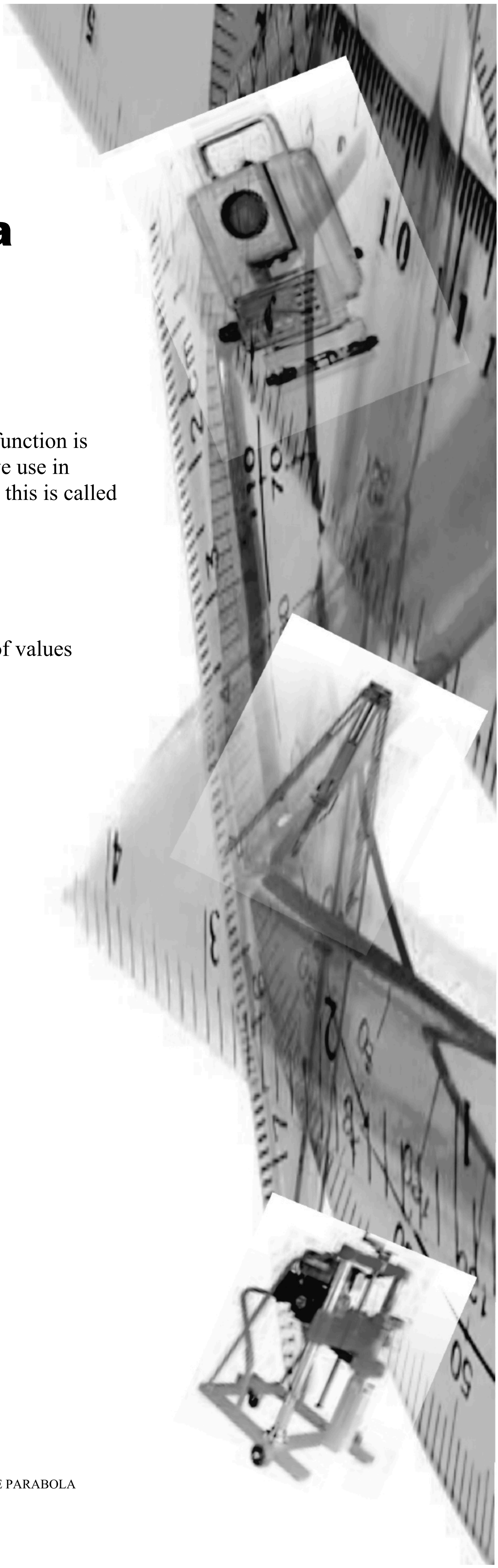
LESSON 6

Graphs: The Parabola



About this lesson

Many relationships can be drawn as a graph, where the function is written like this $y = x^2$. There is also another notation we use in mathematics as we begin to work with more graphs, and this is called 'function notation'.

In this lesson you will:

- draw graphs of a quadratic function using a table of values
- sketch the quadratic function $y = x^2 + q$
- sketch the quadratic shift function $y = (x - p)^2$
- sketch $y = ax^2$

Function notation

In function notation, $y = x^2$ is written as $f(x) = x^2$, indicating that y is a function of x. In other words as x changes, so the y value changes.

For example: If we want to refer to the y value when $x = 1$ for the function $f(x) = x^2$, we write $f(1)$.

Therefore, $f(1) = (1)^2 = 1$. This means that when $x = 1$, $y = 1$ in the function $f(x) = x^2$.

$f(-3) = (-3)^2 = 9$ This means that when $x = -3$, $y = 9$ in the function $f(x) = x^2$.

$f(2) = (2)^2 = 4$ This means that when $x = 2$, $y = 4$ in the function $f(x) = x^2$.

We know that all graphs of linear functions are straight lines. What about graphs of quadratic functions?

Perhaps it is easier to introduce shapes of quadratic functions through an example.

Example

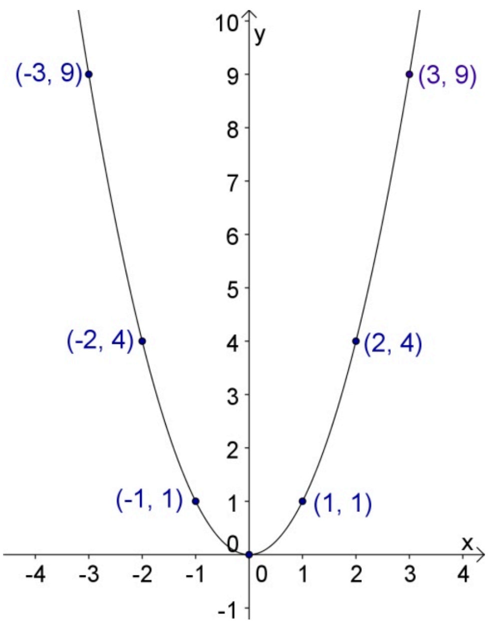
Graph 1

When there is some kind of a relationship between y and x, for example, in $y = x^2$, we say that y is a function of x. In the same way, when we have $d = t^2$, we say the distance, d, is a function of time, t.

In some cases you may see this written: $f(t) = t^2$

Let us use the table of co-ordinates to get the points that we can join to draw this graph.

x	− 3	− 2	− 1	0	1	2	3
y	9	4	1	0	1	4	9



The graph of $y = x^2$ is a curve that has co-ordinates of the form $(x; x^2)$. It turns at point (0; 0). Using the standard form $y = ax^2 + bx + c$, we see that in the equation $y = x^2$, $a = 1$, $b = 0$ and $c = 0$.
We can see that if $c = 0$, the graph will go through (0; 0).

The turning points of these graphs are rounded. They are not sharp. They are like the bends of a road.

*This graph is called a **parabola**. All graphs of quadratic functions are parabolas.*

Shifts of graphs of quadratic functions

The graph of $y = x^2$ is a good way to start understanding graphs of second degree functions. What about changes from this graph? We saw the shifts and other changes in straight line graphs. Let us look at such changes for a parabola.

First, let us look at graphs of the form $y = x^2 + q$, where q stands for any number. Here we see that $x^2 + q$ is the same as $y + q$, since our first graph was $y = x^2$. This information is important, because we can say that each pair of co-ordinates of the new graph, $y = x^2 + q$, will take the new form $(x; x^2 + q)$.

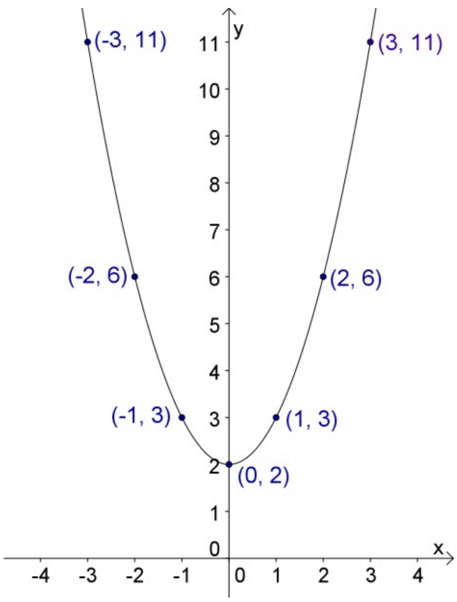
Look at these examples.

Example

Graph 2: $y = x^2 + 2$

The table for this graph will look like this:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
x^2+2	11	6	3	2	3	6	11

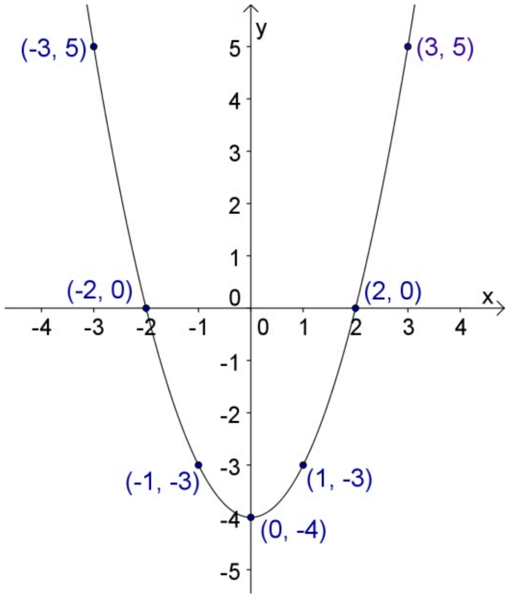


The curve of $y = x^2 + 2$ can be found by adding 2 to all the values of the y –coordinates of $y = x^2$. The values of the x –coordinates do not change. The two graphs are of the same shape exactly. $y = x^2 + 2$ is the graph of $y = x^2$ moved up along the y –axis 2 units. This is a shift upwards.

Example

Graph 3: $y = x^2 - 4$

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 - 4$	5	0	-3	-4	-3	0	5



The curve of $y = x^2 - 4$ can be found by subtracting 4 from all the values of the y –coordinates of $y = x^2$. Therefore, the graph of $y = x^2 - 4$ is the graph of $y = x^2$ moved downwards along the y –axis 4 units.

Now draw these graphs of parabolas on your own.

ACTIVITY 1

Draw the following graphs:

1. $y = x^2 + 3$

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2. $y = x^2 - 3$

Try a rough sketch first, showing that you have an idea of what is likely to happen.

Observations: remarks made after looking carefully at something.

What observations can you make about the shifts of the graphs from $y = x^2$? Are there any general observations that you can make about any graph of the form $y = x^2 + q$? State clearly what happens when $q > 0$ and when $q < 0$.

$>$ means "greater than", and $<$ means "less than"

If $q > 0$, the graph is shifted upwards q units. The x –coordinates remain the same.
If $q < 0$, the graph is shifted downwards q units. The x –coordinates remain the same.

ACTIVITY 2

Make (sketches) rough drawings of the following graphs.

1. $y = x^2 + 1$

2. $y = x^2 - 5$

3. $y = x^2 - 4$

4. $y = x^2 - 2,5$

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The above examples are graphs of the form $y = x^2 + q$. What about the graph of $y = ax^2 + bx + c$?

Graphs of $y = ax^2 + bx + c$

Let us look at a simple example: $y = (x-2)^2$

We can expand this out as follows $y = (x-2)^2 = x^2 - 4x + 4$
Here we see that $a = 1$, $b = -4$ and $c = 4$.

To help us recognise the shifts made from the graph of $y = x^2$, let us look at the examples $y = (x-2)^2$, and $y = (x+2)^2$ together.

Examples

Graph 4: $y = (x-2)^2$

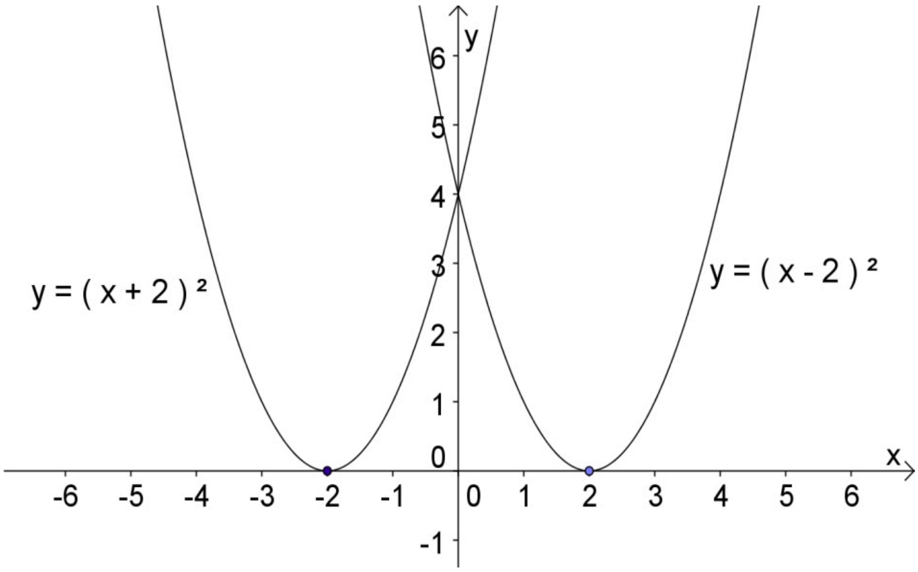
We will use the table of coordinates again.

x	–3	–2	–1	0	1	2	3	4	5	6
$x-2$	–5	–4	–3	–2	–1	0	1	2	3	4
$(x-2)^2$	25	16	9	4	1	0	1	4	9	16

Graph 5: $y = (x+2)^2$

The table of coordinates will look like this:

x	–5	–4	–3	–2	–1	0	1	2	3
$x-2$	–3	–2	–1	0	1	2	3	4	5
$(x+2)^2$	9	4	1	0	1	4	9	16	25



When we compare these graphs with the graph of $y = x^2$, we see the following:

The graph of $y = (x-2)^2$ is the graph of $y = x^2$ shifted 2 units to the right along the x –axis. The values of the y –coordinates do not change.

The graph of $y = (x+2)^2$ is the graph of $y = x^2$ shifted 2 units to the left. Again the values of the y –coordinates do not change.

The above drawing is a drawing of the two graphs on the same system of axes. Drawings of more than one graph are made on the same axes in order to compare the graphs.

Same system of axes: *drawing of more than one graph are made on the same axes.*

You can use the same method in the following activity.

ACTIVITY 3

Draw the following graphs.

1. $y = (x-3)^2$

2. $y = (x+3)^2$

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What observations can you make about the shifts of the graphs from $y = x^2$? Are there any general observations that you can make about any graph of the form $y = (x-p)^2$ for any value of p? State clearly what happens when $p > 0$ and when $p < 0$.
Remember, $x-(-p) = x+p$.

From the examples we can make the following general observations:

- When $p > 0$ the graph of $y = (x - p)^2$ is the same as the graph of $y = x^2$ shifted p units to the right.
- When $p < 0$ the graph of $y = (x - p)^2$ is the same as the graph of $y = x^2$ shifted p units to the left.

The graph of the function $y = (x - p)^2 + q$

Now we must combine the two types of graphs we have discussed so far. These graphs are $y = x^2 + q$ and $y = (x - p)^2$ All quadratic graphs of the form $y = ax^2 + bx + c$ can be written as a combination of these two graphs, in the form $y = (x - p)^2 + q$.

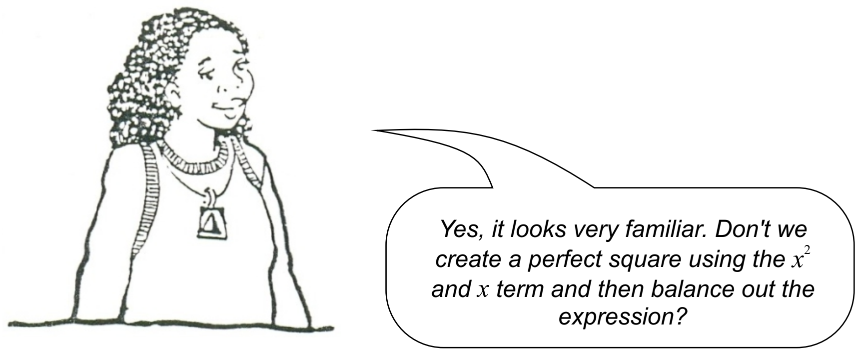
If we write $y = x^2 - 6x + 7$ in the form $y = (x - 3)^2 - 2$, this makes it easier for us to draw the graph of this function. We can easily say what the shape of the function is going to be.

Let us look at this example.

Example

$$\begin{aligned} y &= (x - 3)^2 - 2 \\ &= x^2 - 6x + 9 - 2 \\ &= x^2 - 6x + 7 \end{aligned}$$

So we see that $y = (x - 3)^2 - 2 = x^2 - 6x + 7$. Can you remember how we would get from $x^2 - 6x - 7$ to $(x - 3)^2 - 2$?



That's correct. If you can't remember, check back through the activities in Lessons 4 and 5. It's called completing the square.

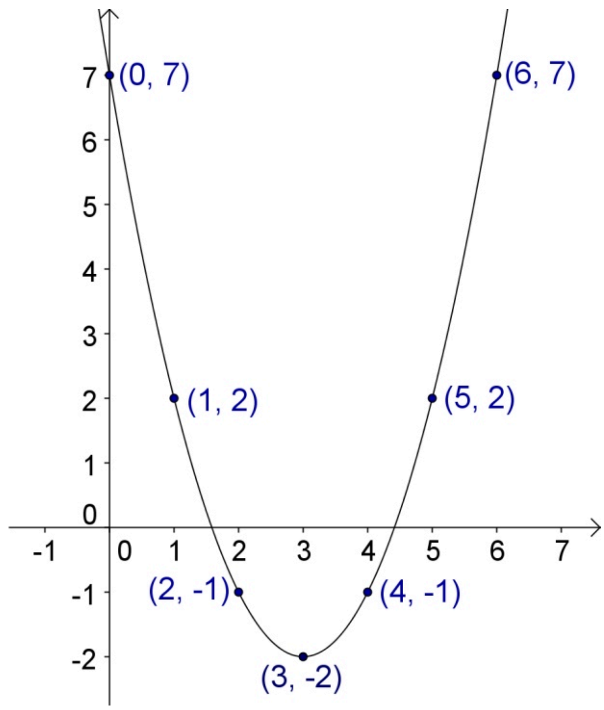
Using the first two terms, $x^2 - 6x$, we determine what we would need to create a perfect square. To find this value, we halve the coefficient of x and square it. In this example, that would be $\left(\frac{1}{2} \text{ x } 6\right)^2 = 9$. So we need to add 9 to $x^2 - 6x$ to make it a perfect square. Having added 9, we need to subtract it again so that we don't change the value of the expression.
$y = x^2 - 6x + 9 - 9 + 7$ then becomes $y = (x - 3)^2 - 2$. This is achieved by factorising the first three terms and then simplifying the last two terms.

What will be the shape of the graph of $y=(x-3)^2-2$?

In this case we know that the function will be like the graph of $y=x^2$ shifted 3 units to the right along the x –axis and shifted 2 units downwards along the y –axis. This very important form of writing a quadratic function helps us sketch the graph. Let us test this statement.

Let us draw $y=x^2-6x+7$ using the table of co-ordinates.

x	–1	0	1	2	3	4	5	6
x^2	1	0	1	4	9	16	25	36
$-6x$	6	0	–6	–12	–18	–24	–30	–36
$+7$	7	7	7	7	7	7	7	7
x^2-6x+7	14	7	2	–1	–2	–1	2	7



This is a combination of the shifts of two graphs, $y=(x-3)^2$ and $y=x^2-2$. The graph of $y=(x-3)^2-2$ moves 3 units to the right, along the x –axis, and 2 units downwards, along the y –axis.

ACTIVITY 4

Draw the graph of $y=(x+3)^2-4$. Use the table of co-ordinates. What observations can you make about the shifts of this graph? Can you make any observations about the shifts of graphs of the form $y=(x-p)^2+q$? Say what happens to the graph when p is positive or negative and what happens when q is positive or negative?

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Although quadratic functions are usually presented in the form $y=ax^2+bx+c$, for drawings many people like to use the form $y=y=(x-p)^2+q$.

Let's have another look at how we get the equation of the parabola from the form $y=x^2+bx+c$ into $y=(x-p)^2+q$.

Consider the parabola $y=x^2+4x+3$. Let's try to put that into the form $y=(x-p)^2+q$. Having the parabola in this format will help us to see what shift has taken place.

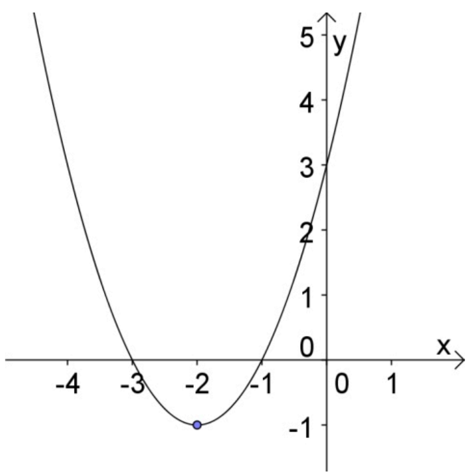
$y=x^2+4x+3$

$\therefore y=x^2+4x+4-4+3$

We halve the coefficient of x and square it to find the value needed to create a perfect square. Half of 4 is 2 which is then squared to get 4. We add 4 to create a perfect square trinomial and then subtract it again to keep the value of the original expression.

$\therefore y=(x+4)^2-1$

We have factorised the first three terms and simplified the last two. Now that the parabola is written in the form $y=(x-p)^2+q$, we can see that the graph $y=x^2$ has been shifted 4 units to the left along the x –axis and 1 unit down along the y –axis.



ACTIVITY 5

Change the following parabola equations from the form $y = ax^2 + bx + c$ into the form $y = (x - p)^2 + q$.

1. $y = x^2 + 8x + 15$

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2. $y = x^2 - x + 3$

All the examples that we have used so far have $a = -1$. What happens when a is negative?

Graphs of $y = ax^2 + bx + c$ when $a < 0$

What do you think happens when $a = -1$? Try these functions.

ACTIVITY 6

Draw the following graphs using a table of values:

1. $y = -x^2$
2. $y = -x^2 + 3$

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3. $y = -(x - 2)^2 + 3$

Say how these graphs are different from the graph of $y = x^2$.

An obvious question to ask will be: What happens when a is not equal to 1 or -1? Well, let us look at that question now.

Graphs of $y = ax^2 + bx + c$ when $a \neq 1$ and $a \neq -1$

We should discuss this question through an activity, since you already know how to draw graphs of quadratic functions. Let us take the simplest graph of $y = ax^2$ and see what happens to it if a is not 1 or -1.

ACTIVITY 7

Using tables, draw the following graphs on the same system of axes.

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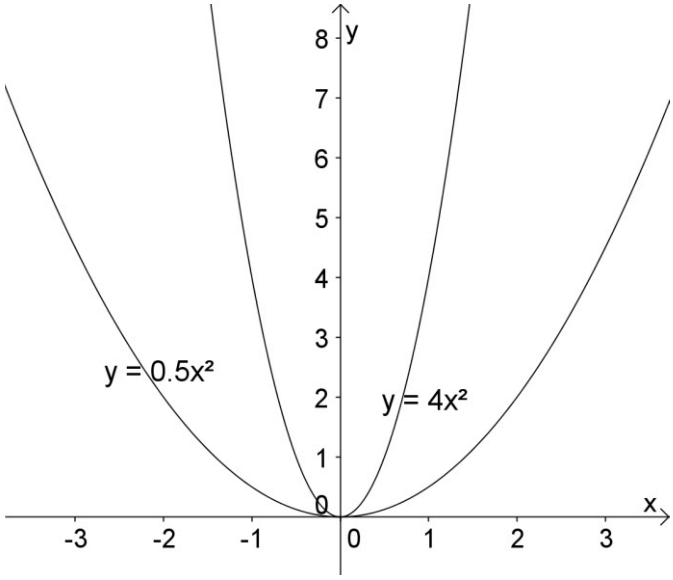
1. $y = x^2$ 2. $y = \frac{1}{2}x^2$ 3. $y = 2x^2$ 4. $y = 4x^2$

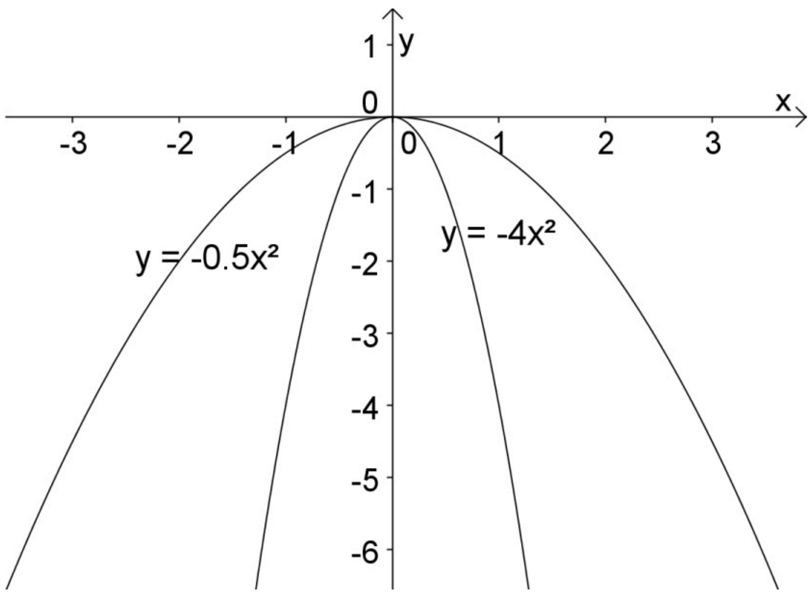
What have you noted from the drawings?

When $0 < a \leq 1$, the graph of $y = ax^2$ flattens out. It becomes a wider 'cup'.
When $a > 1$, the graph of $y = ax^2$ gets narrower. The bigger a is, the steeper the graph.



The same thing happens when a is negative. The only difference is that all the graphs will face downwards.

This is shown below.



Why quadratic graphs?

Why do we have to graph quadratic functions? We use graphs to help solve quadratic equations, like finding roots of an equation. Do you remember how we solved quadratic equations of the form $ax^2 + bx + c = 0$? We said the method is called finding roots of equations. Roots are those values of *x*, the variable, which will make the expression equal to zero or which satisfy the equation.

The questions may be as follows: 'Find roots of an equation,' or: 'For which values of *x* is the function equal to zero?' Both questions mean the same thing.

There are many processes that can be expressed in terms of quadratic functions. In our workplaces we are sometimes given charts or graphs to use. We are expected to be able to read or draw graphs of many processes at work. It may not be very clear when you look at these different processes, but the difference from one process to another may be the value of *a*, *b* or *c*. The following activity will give you an idea of one of the many examples.

ACTIVITY 8

When a car accident occurs traffic officers know that the collision impact of the car is a function of its speed. The collision impact of the car is the result of the force that a car experiences when it smashes against something. For a certain car, the formula is $I = 2v^2$.

In this formula *I* stands for the collision impact and *v* stands for the speed in *km/h*.

1. Draw the graph of this relationship.

2. What is the collision impact of the car at 30 km/h and at 50 km/h?

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Quadratic graphs in physics

Quadratic graphs are used in physics to look at the motion (movement) of objects. When a ball is thrown up in the air with an initial speed of 10 metres per second, the height of the ball above the ground at any given moment is $h = 10t - 4,9t^2$ where *h* is the height, and *t* is the time after the throw.

What is the height of the ball above the ground after 1 second, and after 2 seconds?

According to your estimation, after how many seconds does the ball come back to the position from which it was thrown?

This type of investigation is used when people are calculating movement of objects like spaceships.

Another use of graphs of quadratic equations is in economics. When plans are made about prices of a certain product, graphs are drawn to give a general picture of how the relationship between the supply and demand of that product will affect the price.

Summary

In this lesson you looked at a quadratic function and said what the graph looks like. The following self-check exercise will give you an idea of how well you understand this lesson.

CHECKLIST

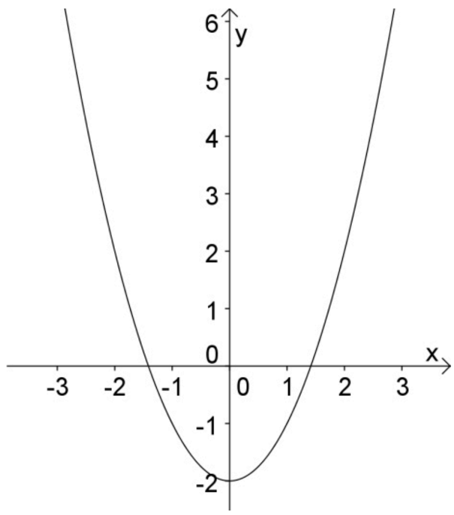
Are you able to:

- draw the graph of a quadratic function using a table of values
- draw a rough sketch of the quadratic function $y = x^2 + q$ by recognising the effect that the q has on $y = x^2$
- draw a rough sketch of the quadratic function $y = (x - p)^2$ by recognising the effect that the p has on $y = x^2$
- draw a rough sketch of $y = ax^2$ by recognising the effect that the a has on $y = x^2$.
- identify the use of graphs of a quadratic function.

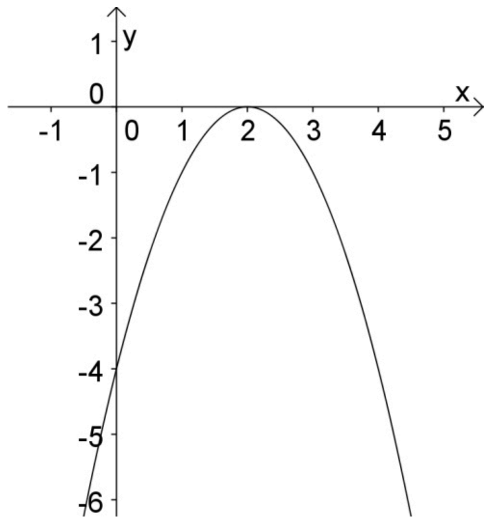
SELF-CHECK EXERCISE

1. Write down the functions of the given graphs.

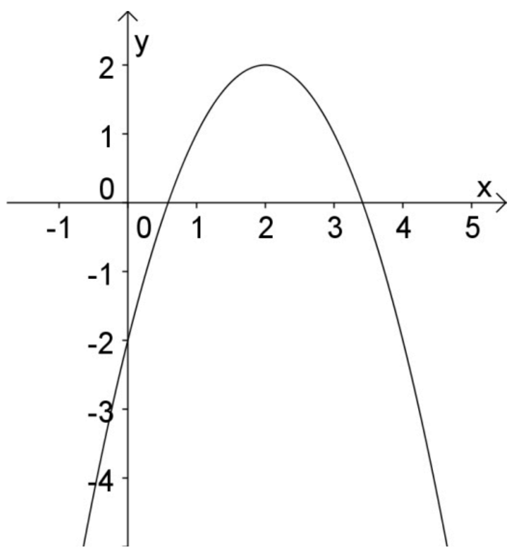
a.



b.



c.



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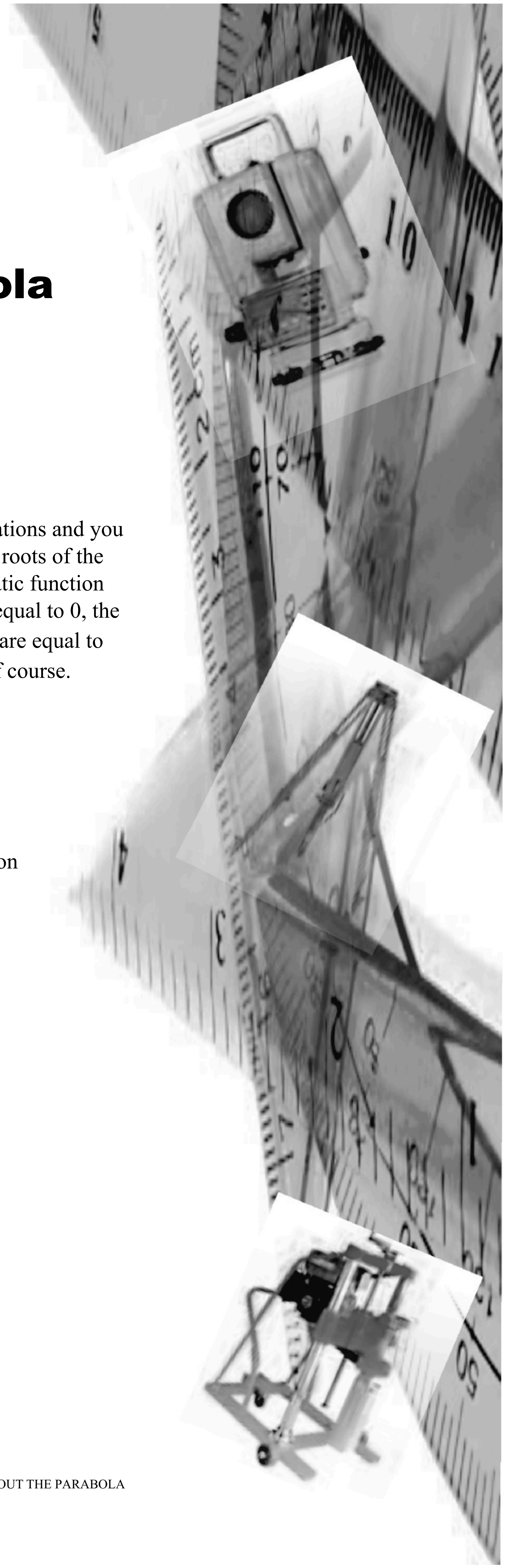
2. Draw the graph of $y = (x-1)^2 - 4$.

LESSON 7

Graphs:
More about the Parabola



About this lesson

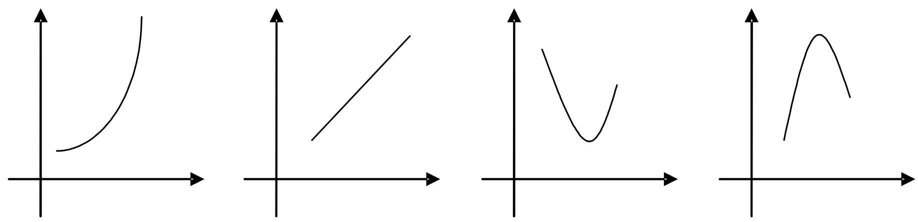
In the last lesson you learnt how to solve quadratic equations and you discovered that the solution of a quadratic gave you the roots of the equation. Do you remember where the roots of a quadratic function are shown on the graph? Since we make the value of y equal to 0, the roots are the points on the graph where y –co-ordinates are equal to zero. Do you remember where this is? On the x –axis of course.

In this lesson you will:

- draw quadratic function graphs
- calculate the axis of symmetry of a quadratic function
- find the turning point
- sketch parabolas
- apply quadratic functions to practical situations

Graphs: Introduction to quadratic functions

Example 1



A B C D

Which of the graphs A - D best fits each of the following descriptions?

i) Violence was falling but is now rising
ii) Inflation, which rose slowly until 1980 is now rising rapidly
iii) Unemployment, which has been rising steadily is now falling
iv) The price of cigarettes has been rising steadily over the last two years

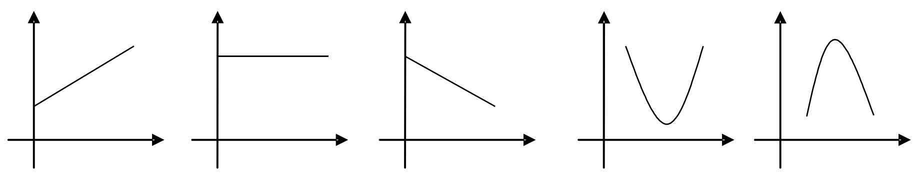
Solution

i) C
ii) A
iii) D
iv) B

With graphs such as these we are able to express the relationships between variables, for example, the rate over time in graph A. There are two factors involved, the time and the rate. We call the two factors variables.

ACTIVITY 1

The following graphs show how a variable y changes as x changes. The x is on the horizontal axis, and the y is on the vertical axis.



A B C D E

From the graphs, choose the graph which, as x increases,

1. y increases
2. y decreases
3. y remains constant
4. y first increases then decreases
5. y first decreases then increases

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Do you see any parabolas in these graphs?

Yes, graphs d and e are parabolas.

A quadratic function has a general equation of the form $y = ax^2 + bx + c$ where a, b and c stand for constants and $a \neq 0$.

That's right. These are the graphs which represent quadratic equations.

ACTIVITY 2

1. Plot the graphs of the following quadratic functions. Calculate enough points to establish a pattern then connect them to form the curve. Use the same intervals for all six graphs, although the intervals on the y - axis need not be the same as those on the x - axis.

   a) $y = x^2$

   b) $y = \frac{1}{2}x^2$

   c) $y = 2x^2$

   d) $y = -\frac{1}{2}x^2$

   e) $y = x^2 - 9$

   f) $y = x^2 - 2x - 8$

2. What is the effect on the graph of:

   a) changing the sign of the x^2 coefficient?
   b) adding an x term?
   c) adding a constant term?

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Quadratic function graphs

In Activity 2 you plotted the graphs of several quadratic functions. Each graph was a $\cup$ - shaped figure called a parabola.

From the graphs you should have reached the following conclusions:

1. The constant a tells us what shape the parabola will be.

   a) If $a > 0$ the graph opens upward (smiles), e.g. $y = 2x^2$

   b) If $a < 0$ the graph opens downward (frowns), e.g. $y = -\frac{1}{2}x^2$

   c) As the a gets closer to 0, the graph gets flatter almost becoming a horizontal line. This is not surprising because when $a = 0$ we have a linear function such as $y = x + 2$.

2. The constant c equals the y - intercept since $y = c$ when $x = 0$.

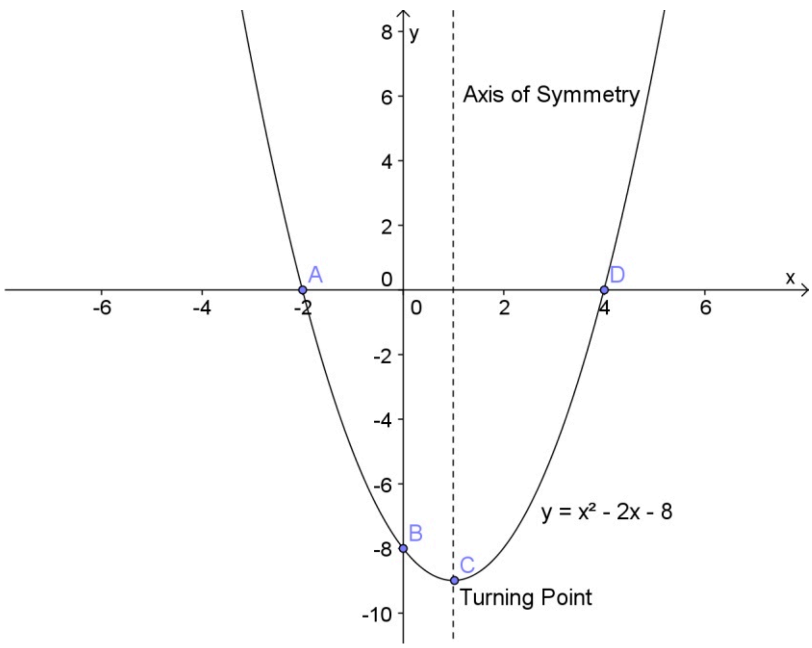
3. The constant b affects the position of the parabola on the Cartesian plane.

In Activity 2 you drew the graph of $y = x^2 - 2x - 8$. The lowest point is called the turning point (minimum point). If the parabola was frowning, $(a < 0)$, then the turning point would be the highest point (maximum point).

The turning point is the most important point on the graph. With the turning point, you can sketch a fairly good parabola with very little other information.

If you fold your page along the axis of symmetry the two sides of the graph should match exactly. So the axis of symmetry is the centre line.

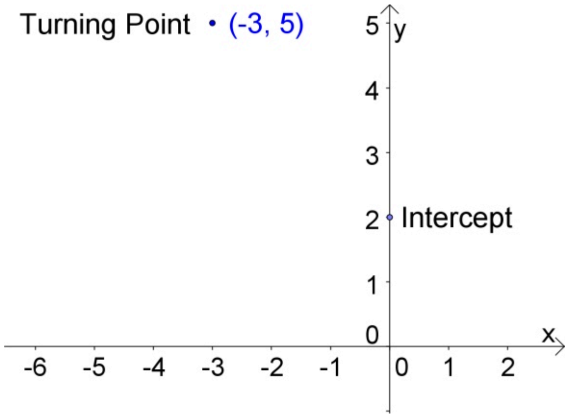
The vertical line through the turning point is called the axis of symmetry.



Example 2

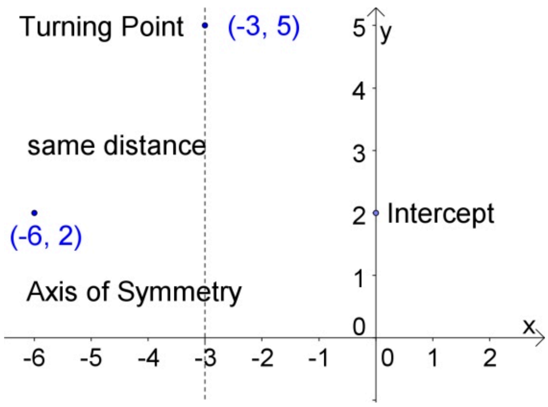
Suppose the turning point is (-3; 5) and the y –intercept is 2, what would the graph of the quadratic function look like?

Solution

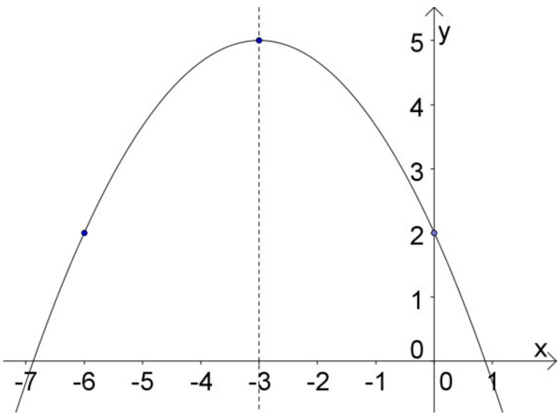


As you see on the left hand side,

1. Plot the turning point and the y - intercept



2. then draw the axis of symmetry through the turning point. Since this is the axis of symmetry there will be a third point which is opposite the y - intercept and the same distance from the dotted line.



3. You have three points and a fair idea of how a parabola should look. You can now draw a reasonably good graph of the function.

ACTIVITY 3

Sketch the graphs of the following quadratic functions. Each function has a turning point and y –intercept. Remember you do not need to use the table method. You have just learnt a much quicker method.

1.	Turning point: $(2; 3)$	y –intercept: 6
2.	Turning point: $(3; -1)$	y –intercept: -5
3.	Turning point: $(-3; 2)$	y –intercept: -6
4.	Turning point: $(-3; 6)$	y –intercept: -5
5.	Turning point: $(-3; -4)$	x –intercept: 2
6.	Turning point: $(3; 3)$	x –intercept: 1

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The axis of symmetry

We saw in Activity 2 that the quadratic function $y = x^2 - 2x - 8$ had a turning point of (1; –9). We were able to locate this point using the table method. We have also learnt how important the turning point can be in plotting graphs quickly. You will have noticed that the axis of symmetry goes through the x –value of the turning point therefore the equation of the axis of symmetry for the graph $y = x^2 - 2x - 8$ must be $x = 1$.

This method of finding the axis of symmetry is long and tedious so we have established a formula. The axis of symmetry of a quadratic function is $x = \dfrac{-b}{2a}$

Example 3

Using the formula, find the axis of symmetry for the quadratic function $y = x^2 - 2x - 8$

Solution

$y = x^2 - 2x - 8 \qquad \therefore a = 1, b = -2, \text{and } c = 8$

$\therefore$ Axis of symmetry is $\quad x = \dfrac{-(-2)}{2(1)}$

$\therefore x = 1$

As you can see from the graph above, this is perfectly true. Let's practice some examples to make sure we can use the formula correctly.

ACTIVITY 4

Find the axis of symmetry of the following parabolas using the formula:

1. $y = 2x^2 + 4x + 6$
2. $y = -x^2 + 4x - 4$
3. $y = x^2 - 4$

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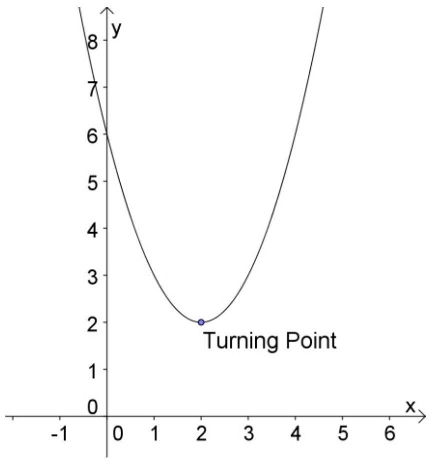
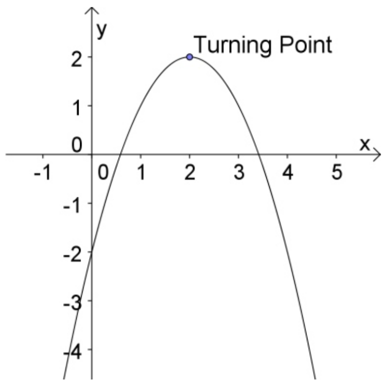
Now let's take a closer look at the turning point of the parabola.

The turning point





The curve travels in one direction, then, when it reaches a minimum or a maximum point it changes direction. This is called the turning point (TP). At the turning point, the x –value is the same as the axis of symmetry $x = \frac{-b}{2a}$. If you can find the y - value when $x = \frac{-b}{2a}$, then you'll have the turning point.

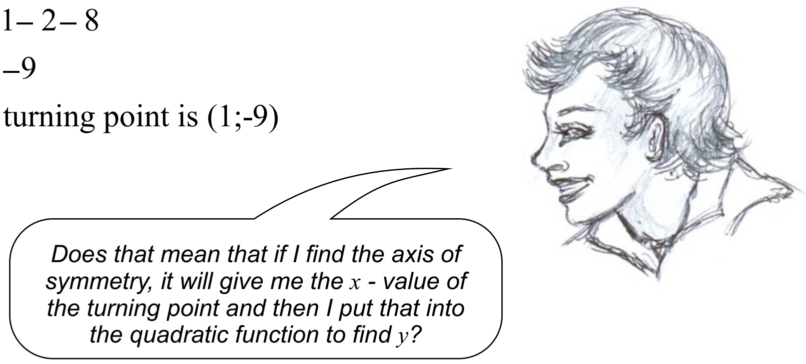
Let's take another look at example 3. Using the quadratic $y = x^2 - 2x - 8$, you were asked to find the axis of symmetry which was $x = 1$.
We know that this is the x –value of the turning point so how do we calculate the y - value? Well that's fairly easy; we can substitute $x = 1$ into $y = x^2 - 2x - 8$.

$y = (1)^2 - 2(1) - 8$ (substitute $x = 1$ into given quadratic function)

$\therefore y = 1 - 2 - 8$

$\therefore y = -9$

$\therefore$ the turning point is (1;-9)



Yes, it does. Why don't we try some activities?

ACTIVITY 5

Find the turning points of the following parabolas

a) $y = x^2 - 5x + 2$

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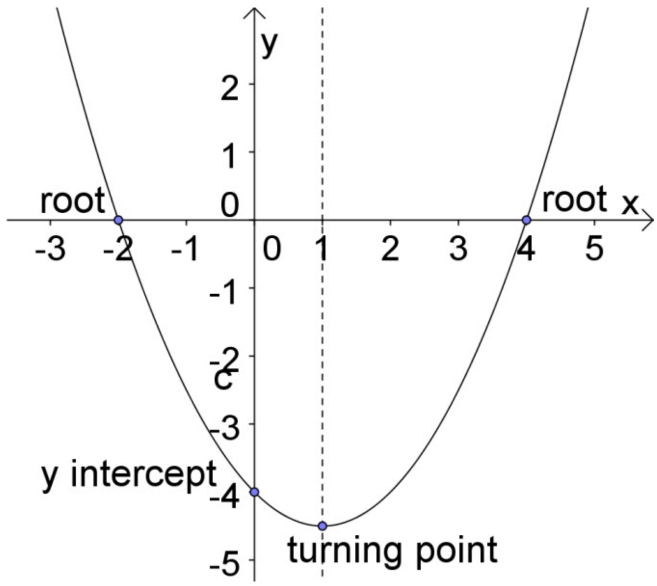
b) $y = 2x^2 + 3x - 1$

Summary of properties of $y = ax^2 + bx + c$



1. If $a > 0$, the graph is $\cup$ and if $a < 0$, the graph is $\cap$.

2. The parabola cuts the y –axis at $y = c$.

3. To calculate the roots of the equation put $y = 0$ i.e. $y = ax^2 + bx + c$ and solve for x.

4. The axis of symmetry is $x = \dfrac{-b}{2a}$

5. Find the y –value at the turning point by substituting the x –value of the axis of symmetry into the quadratic function.

ACTIVITY 6

1. Read over the whole lesson this far from beginning to end.

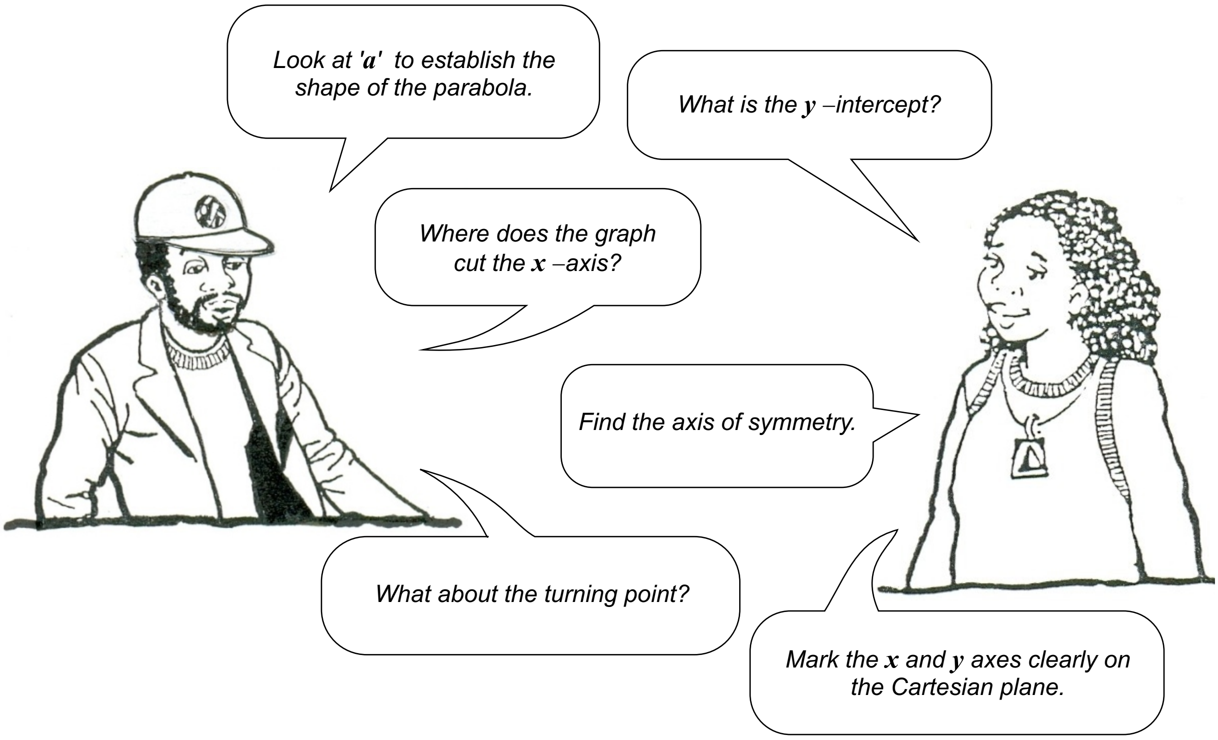
2. Mark the areas you are not yet clear about.

3. Make a list of questions to ask your tutor the next time you see him/her.

Then carry on with the lesson.

Sketching the parabola

Remember there is a pattern of action.



You don't have to fill in all the numbers, only the ones that are important.
The graph does not have to be accurate. Start by factorising if you can.

Example 4

Sketch the graph of $y = -2x^2 - 3x + 5$

Follow the pattern of action.

Solution

> *Be careful,* $y = -2x^2 - 3x + 5$ *is not equal to* $y = 2x^2 + 3x - 5$.
>
> *Do you know why?*

1. a is negative so the graph is $\cap$.

2. $c = 5$ so the graph cuts the y –axis at 5.

3. Put $y = 0$ to find the x –intercepts

$\therefore -2x^2 - 3x + 5 = 0$

$\therefore 2x^2 + 3x - 5 = 0$ (multiply both sides by -1)

$\therefore (2x + 5)(x - 1) = 0$ (factorise trinomial)

$\therefore 2x + 5 = 0 \quad or \quad x - 1 = 0$

$$x = -\frac{5}{2} = -2,5 \quad or \quad x = 1$$

So the graph cuts the x –axis at 1 and $-2,5$.

4. The axis of symmetry is $x = \frac{-b}{2a} = \frac{3}{-4}$

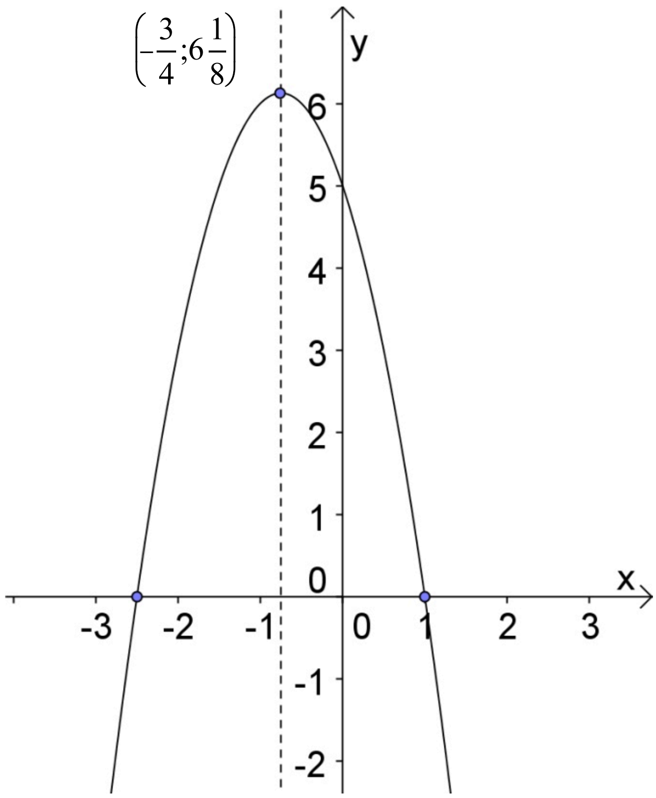
5. The y - value at the turning point is $y = -2\left(-\frac{3}{4}\right)^2 - 3\left(-\frac{3}{4}\right) + 5$

$$= -\frac{9}{8} + \frac{9}{4} + 5$$

$$= 6\frac{1}{8}$$

$\therefore$ TP $\left(-\frac{3}{4};6\frac{1}{8}\right)$

6.



ACTIVITY 7

Draw rough sketches of the graphs of:

1. $y = x^2 - x - 2$

2. $y = -x^2 - 2x + 3$

3. $y = x^2 + 6x - 5$

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4. $y = x^2 - 6x + 9$

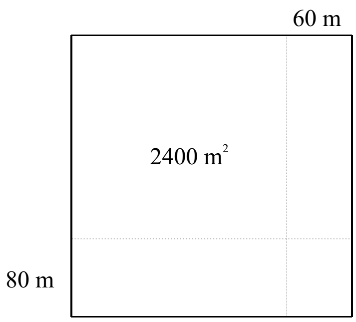
Practical applications of quadratic functions

It is very rare that people think of quadratic equations as solutions to everyday problems. Here are a few examples to show that we can use quadratic equations to solve such problems.

Example 5

A farmer needs to divide up his square plot for each of his sons. He decides to give his youngest son a plot of ground which was divided by decreasing the original plot on one side by 60 m and the other side by 80 m. The area formed is now 2 400m^2. What were the original dimensions of the plot?

Solution



Let's call the original length x, because it is a square the breadth will also be x and the area would be x^2.

Original area $= x^2$ (area of a square) = length < x > breadth)

Dimensions of youngest son's plot: length $= x - 60$
breadth $= x - 80$
area $= 2\,400 m^2$

$\therefore (x - 60)(x - 80) = 2400$

$\therefore x^2 - 140x + 4800 = 2400$

$\therefore x^2 - 140x + 2400 = 0$

$\therefore (x - 120)(x - 20) = 0$

$\therefore x = 120 \quad or \quad x = 20$

We have two values of x, so which is correct? If we put x = 20 then the original plot would be 20 x 20 =400m^2, but we know that the farmer had 2 400m^2 after dividing the farm so this answer is impossible.

If we put x = 120 then the original plot would be 120 x 120=14 400m^2 which is more likely. Therefore the original dimensions of the farm were 120 x 120 = 14 400m^2

ACTIVITY 8

The plan of a house has the shape of a rectangle 21 m long and 10 m wide. Determine the thickness of its walls if the inside area of the house, partitions included, equals 180m^2.

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Summary

This lesson focused on identifying the axis of symmetry and turning point of the graph of a quadratic function, drawing the graph of a quadratic function and using quadratic functions to solve everyday problems.

CHECKLIST

Are you able to:

- draw quadratic function graphs
- calculate the axis of symmetry of a quadratic function
- find the turning point
- sketch parabolas
- apply quadratic functions to practical situations.

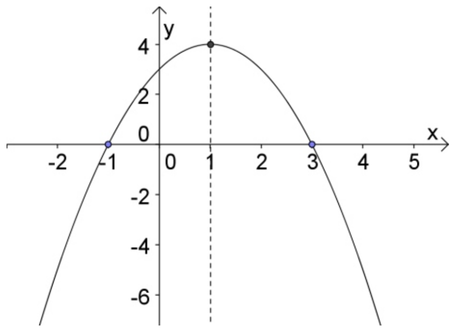
SELF-CHECK EXERCISE

Explain briefly

1. a) the axis of symmetry of a parabola
   b) the turning point of a parabola.

2. Describe how the sign of the value of *a* in $ax^2 + bx + c = 0$ determines the shape of the graph.

3. Here is the graph of $y = -x^2 + 2x + 3$



   Answer the following questions:

   a) What line is the axis of symmetry?
   b) Where does the graph cut the y–axis?
   c) What is the value of $-x^2 + 2x + 3$ when $x = 0$ and when $x = 1$?
   d) What is the maximum $-x^2 + 2x + 3$ value can have?
   e) What are the roots of the equation?
   f) What are the co-ordinates of the turning point?

4. Draw a rough sketch of the graph $y = 2x^2 - 7x + 6$, using the properties of the parabola.

5. Mirror tiles are used to cover a certain rectangular area of a dining room wall. There are ten tiles in each row and three rows. If the area covered is to be doubled by increasing the length and the width by the same number of tiles, how many tiles will there be in each row and how many rows will there be?

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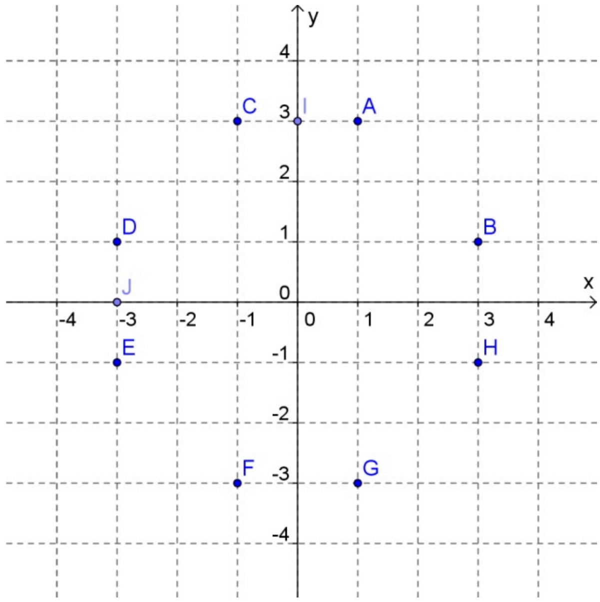
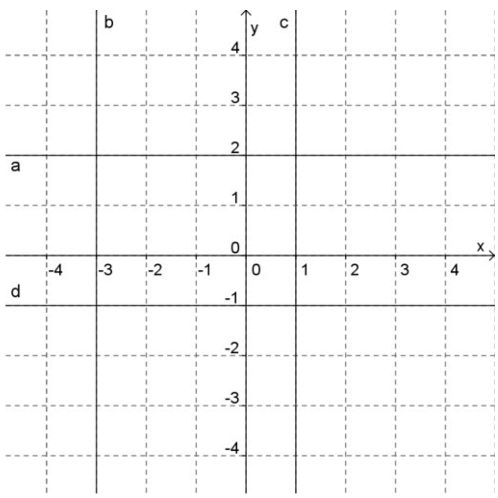
Feedback to Activities

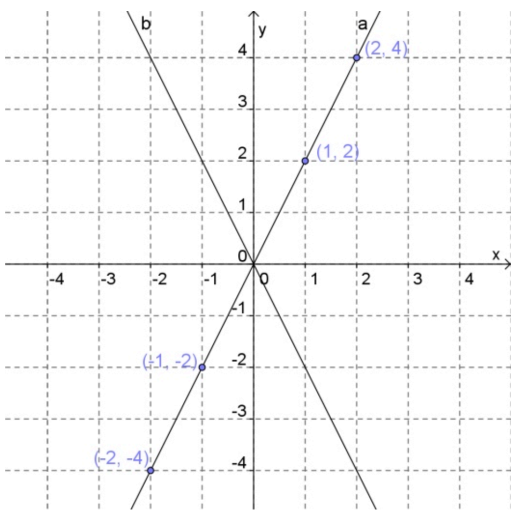
Lesson 1

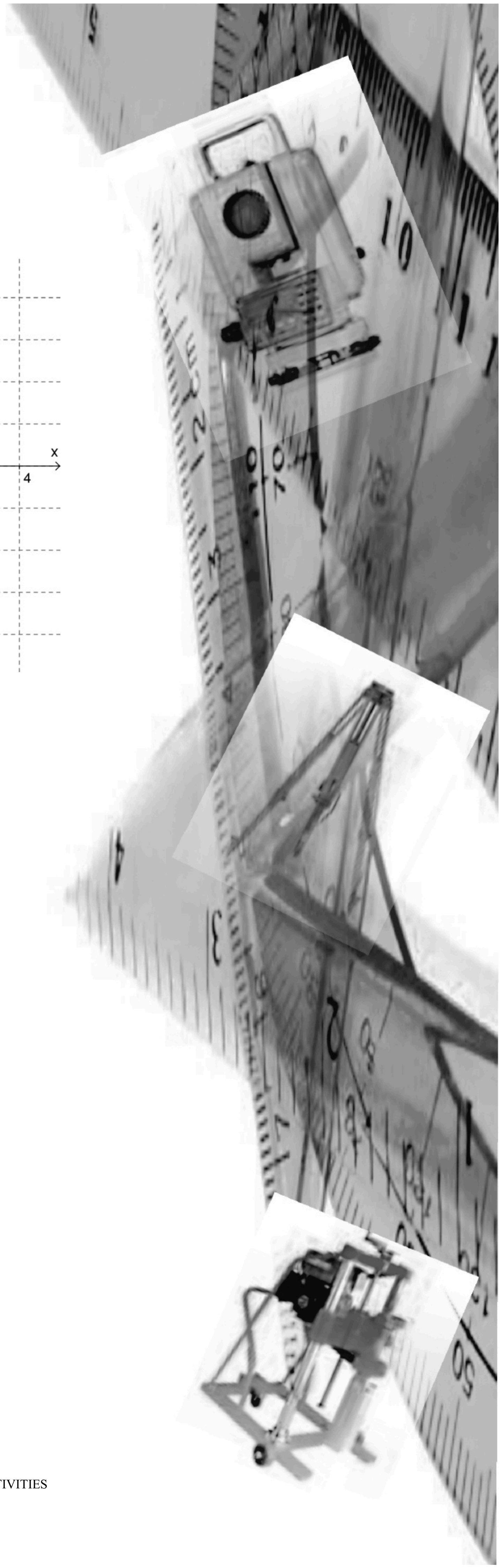
Activity 1



Activity 2







Activity 3

A horizontal line has a slope of zero.

A vertical line has no slope.

Remember that *m* is the notation for gradient and the word *slope* is sometimes used instead of gradient.

1. $m = \frac{7-3}{2-1} = \frac{4}{1} = 4$

2. $m = \frac{5}{-5} = 1$

3. $m = \frac{0}{-4} = 0$ this is the line $y = -3$ which is a horizontal line with a gradient of zero.

4. $m = \frac{1}{-1} = -1$

5. $m = \frac{4}{0}$ which is undefined since we are dividing by zero. This is the vertical line $x = 2$ which has no slope.

Activity 4

The points A, B and C are in a straight line if the gradient from A to B is the same as the gradient from B to C.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

1. $m_{AB} = \frac{3-0}{1-(-2)} = \frac{3}{3} = 1$ and $m_{BC} = \frac{2-3}{0-1} = \frac{-1}{-1} = 1$

   $\therefore\ m_{AB} = m_{BC}$ $\therefore$ the points are in a straight line.

2. $m_{PQ} = \frac{12-6}{8-4} = \frac{6}{4} = \frac{3}{2}$ and $m_{QR} = \frac{18-12}{12-8} = \frac{6}{4} = \frac{3}{2}$

   $\therefore\ m_{PQ} = m_{QR}$ $\therefore$ the points are in a straight line.

3. $m_{LM} = \frac{10-3}{7-(-5)} = \frac{7}{12}$ and $m_{MN} = \frac{14-10}{9-7} = \frac{4}{2} = 2$

   $\therefore\ m_{LM} \neq m_{MN}$ $\therefore$ the points are **not** in a straight line.

Activity 5

1. $y = 2(0) + 3$

   $= 3$

   $\therefore$ (0;3) is on the line

2. $y = -4\left(\frac{1}{4}\right) + 3$

$= -1 + 1$

$= 0$

$\therefore \left(\frac{1}{4};0\right)$ is on the line

3. $y = (-2)^2 + (-2) + 2$

$= 4 - 2 + 2$

$= 4$

$\therefore$ $(-2;8)$ is not on the line

Lesson 2

Activity 1

1. If $m = 2$, $x_1 = -2$ and $y_1 = 4$

$y - 4 = 2(x - (-2))$ (Substitute $x_1 = -2$ and $y_1 = 4$)

$\therefore y - 4 = 2(x + 2)$

$\therefore y - 4 = 2x + 4$ (Expand the right hand side)

$\therefore y = 2x + 8$ (Simplify the equation)

2. If $m = 1$, $x_1 = -2$ and $y_1 = -3$

$y - (-3) = 1(x - (-2))$ (Substitute $x_1 = -2$ and $y_1 = -3$)

$\therefore y + 3 = 1(x + 2)$

$\therefore y + 3 = x + 2$

$\therefore y = x - 1$

3. $m = -1$, $x_1 = 3$ and $y_1 = -5$

$y - (-5) = -1(x - 3)$

$\therefore y + 5 = -1(x - 3)$

$\therefore y + 5 = -x + 3$

$\therefore y = -x - 2$

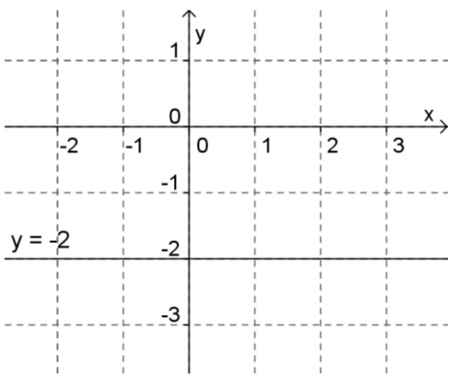
4. $m = 0$, $x_1 = -5$

$y - (-2) = 0(x - (-5))$

$\therefore y + 2 = 0(x + 5)$

$\therefore y + 2 = 0$

$\therefore y = -2$

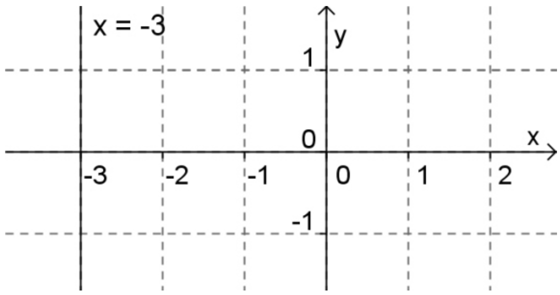


We know that a line with a gradient of zero is a horizontal line and all horizontal lines have an equation of y = something!

5. The equation $y_2 - y_1 = m(x_2 - x_1)$ is not useful here because m is meaningless.
Since the line is vertical and parallel to the y - axis, the equation is $x = -3$

A horizontal line has zero slope.

A vertical line has no slope.



Activity 2

Be careful:

$y - y_1$ *when* $y_1 = -5$
$= y - (-5)$
$= y + 5$

1. If $(3;-5)$ is $(x_1;y_1)$ and $(-1;3)$ is $(x_2;y_2)$, then

$$m = \frac{3-(-5)}{-1-3}$$
$$= \frac{3+5}{-4}$$
$$= -2$$

Now that we know that $m = -2$, we can use $(3;-5)$ for our substitution.

$y - (-5) = -2(x-3)$ (Substitute $x_1 = 3$ and $y_1 = -5$)
$\therefore y + 5 = -2x + 6$ (Expand and simplify)
$\therefore y = -2x + 1$

2. $$m = \frac{2-7}{-2-4}$$
$$= \frac{-5}{-6}$$
$$= \frac{5}{6}$$

$\therefore m = \frac{5}{6}, \quad x_1 = 4, \quad y_1 = 7$

$\therefore y - 7 = \frac{5}{6}(x-4)$

$\therefore y - 7 = \frac{5}{6}x - \frac{20}{6}$

$\therefore y = \frac{5}{6}x + \frac{11}{3}$ or $6y = 5x + 22$

3. $m = \frac{3-3}{1-(-2)}$

$= 0$ This means the line is horizontal.

$\therefore m = 0, \quad x_1 = -2, \quad y_1 = 3$

$\therefore y - 3 = 0(x - (-2))$

$\therefore y - 3 = 0$

$\therefore y = 3$

4. $m = \frac{5-2}{-1-(-1)}$

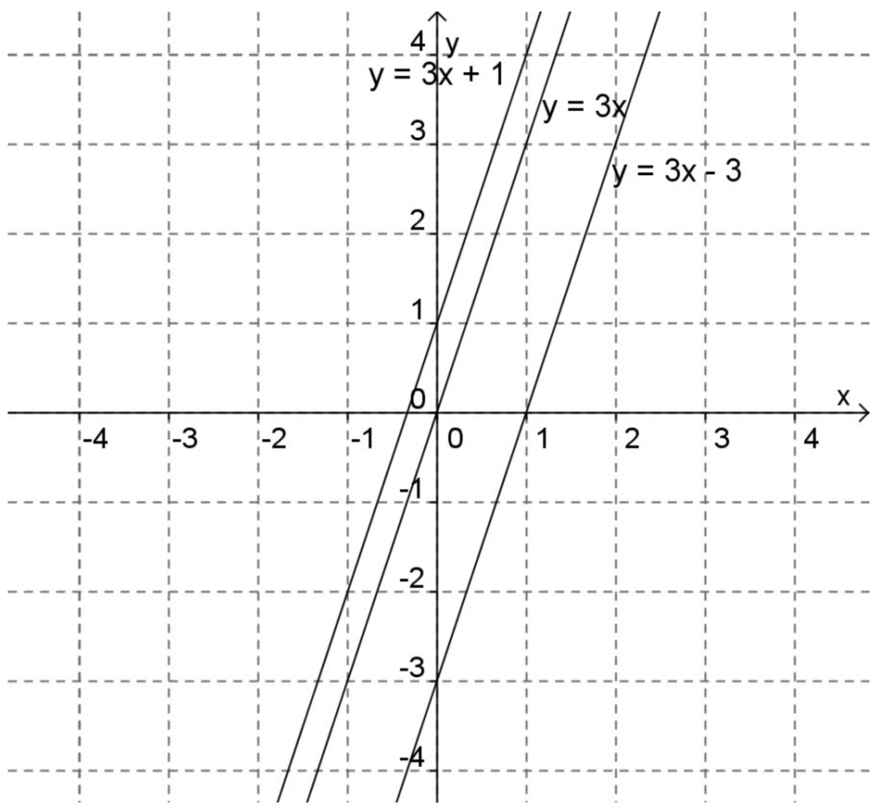
$= \frac{3}{0}$ which is undefined. So this is a vertical line parallel to the y –axis

$\therefore x = -1$

Activity 3

Remember if you want the graphs on the same set of axes, it means that all the lines must go on one Cartesian plane.

So let's draw the graphs and if you are having difficulty, ask your tutor for help.



You should have noticed that all the lines are parallel and they have the same gradient of 3 in each case.

Activity 4

1a. If $(-3;2)$ is $(x_1;y_1)$ and $(1;1)$ is $(x_2;y_2)$, then

$$m=\frac{1-2}{1-(-3)}=-\frac{1}{4}$$

1b. If $(4;3)$ is $(x_1;y_1)$ and $(-1;8)$ is $(x_2;y_2)$, then

$$m=\frac{8-3}{-1-4}=\frac{5}{-5}=-1$$

1c. If $(-3;-5)$ is $(x_1;y_1)$ and $(1;3)$ is $(x_2;y_2)$, then

$$m=\frac{3-(-5)}{1-(-3)}=\frac{3+5}{1+3}=\frac{8}{4}=2$$

Remember switch-rounds?

$$3-4=-(4-3)$$
$$y-x=-(x-y)$$

1d. If $(a;b)$ is $(x_1;y_1)$ and $(b;a)$ is $(x_2;y_2)$, then

$$m=\frac{a-b}{b-a}=\frac{a-b}{-(a-b)}=-1$$

Remember:

|| means parallel
⊥ means perpendicular

2a. $m=-\frac{1}{4}\ \therefore \perp m=4$

2b. $m=-1\ \therefore \perp m=1$

2c. $m=2\ \therefore \perp m=-\frac{1}{2}$

2e. $m=-1\ \therefore \perp m=1$

3a. $m_{AB}=\frac{-2-1}{-4-0}=\frac{-3}{-4}=\frac{3}{4}$ $\qquad m_{CD}=\frac{2-(-1)}{1-(-3)}=\frac{2+1}{1+3}=\frac{3}{4}$

$\therefore m_{AB}=m_{CD}$

$\therefore AB \,||\, CD$

3b. $m_{AB}=\frac{4-(-10)}{0-6}=\frac{14}{-6}=-\frac{7}{3}$ $\qquad m_{CD}=\frac{-3-0}{-4-3}=\frac{3}{7}$

$\therefore m_{AB}\times m_{CD}=-1$

$\therefore AB\perp CD$

3c. $m_{AB}=\frac{1-(-4)}{3-(-2)}=\frac{5}{5}=1$ $\qquad m_{CD}=\frac{3-(-1)}{0-(-5)}=\frac{4}{5}$

Lines are neither parallel or perpendicular

Lesson 3

Activity 1

1. $d=\sqrt{(3-6)^2+(-6-(-2))^2}$

   $d=\sqrt{(-3)^2+(-6+2)^2}$

   $=\sqrt{9+16}$

   $=\sqrt{25}$

   $=5$

2. $d=\sqrt{(0-2)^2+(2-7)^2}$

   $d=\sqrt{4+25}$

   $=\sqrt{29}$

3. $d=\sqrt{(4-0)^2+(0-3)^2}$

   $d=\sqrt{16+9}$

   $=\sqrt{25}$

   $=5$

Activity 2

1. The midpoint of a is $(3;3)$

   The midpoint of b is $(4;-2)$

   The midpoint of c is $(-2;3)$

   The midpoint of d is $(0;-2)$

2. In each case the x –value of the mid-point is the sum of the x –values of the end-points divided by 2 and the y –value of the mid-point is the sum of the y –values of the end points divided by 2.

Activity 3

1. Let $(-2;3)$ be $(x_1;y_1)$ and $(6;3)$ be $(x_2;y_2)$

   $x=\frac{-2+6}{2}$ and $y=\frac{3+3}{2}$

   $=\frac{4}{2}$ $\quad$ $=\frac{6}{2}$

   $=2$ $\quad$ $=3$

   $\therefore$ midpoint is $(2;3)$

2. $\left(1\frac{1}{2};1\right)$

3. $\left(1\frac{1}{2};-4\right)$

Activity 4

1a) At the point of intersection, the y - values of the two lines are the same

$\therefore x+1=-3x+5$

$\therefore x+3x=5-1$

$\therefore 4x=4$

$\therefore x=1$

Now substitute back into either of the two equations.

$y=(1)+1$

$\therefore y=2$

Remember:

$\therefore$ means therefore

$\therefore$ the point of intersection is (1;2)

1b) $2x+3=x-5$

$\therefore x=-8$

Substitute $x=-8$ $\quad y=2(-8)+3$

$\therefore y=-13$

$\therefore$ the point of intersection is $(-8;-13)$

1c) If the equations are not in the standard form of $y=mx+c$, start by changing into that format.

$\therefore 3x+y-2=0 \rightarrow y=-3x+2$

Point of intersection: $x-3=-3x+2$

$\therefore 4x=5$

$\therefore x=\frac{5}{4}$

Substitute $x=\frac{5}{4}$ $\quad y=\left(\frac{5}{4}\right)-3$

$\therefore y=-\frac{7}{4}$

$\therefore$ the point of intersection is $\left(\frac{5}{4};-\frac{7}{4}\right)$

2. First we have to determine the equation of the line through the points (–2;6) and (3; –2).

If (–2;6) is the point $(x_1;y_1)$ and (3;–2) is $(x_2;y_2)$

$$m = \frac{-2-6}{3-(-2)}$$

$$= -\frac{8}{5}$$

Then $y-(-2) = -\frac{8}{5}(x-3)$

$$\therefore y+2 = -\frac{8}{5}x + \frac{24}{5}$$

$$\therefore y = -\frac{8}{5}x + \frac{14}{5}$$

The other equation is $y = 2x - 3$, so we find the point of intersection.

$$2x-3 = -\frac{8}{5}x + \frac{14}{5}$$

$\therefore 10x - 15 = -8x + 14$ (multiply through by 5 to eliminate the fraction)

$$\therefore 18x = 29$$

$$\therefore x = \frac{29}{18}$$

Substitute $x = \frac{29}{18}$ $\quad y = 2\left(\frac{29}{18}\right)x - 3$

$$\therefore y = \frac{2}{9}$$

$\therefore$ the point of intersection is $\left(\frac{29}{18};\frac{2}{9}\right)$

Lesson 4

Activity 1

1. Using the equation $x^2 + y^2 = r^2$, $x = 5$ and $y = -12$

$$r^2 = (5)^2 + (-12)^2$$
$$= 25 + 144$$
$$= 169$$

$\therefore r = 5$ and the equation is $x^2 + y^2 = 169$

2. Cutting the x - axis at 7, $\therefore x = 7$ and $y = 0$

$$r^2 = (7)^2 + (0)^2$$
$$= 49$$

$\therefore r = 7$ and the equation is $x^2 + y^2 = 49$

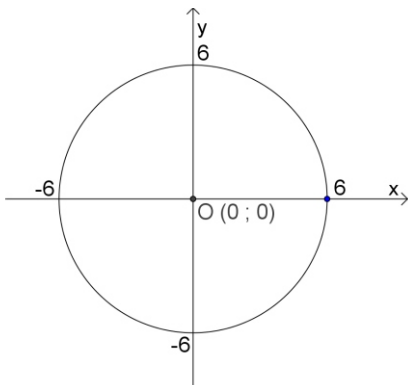
3. Cutting the y - axis at -3, $\therefore\ y=-3$ and $x=0$

$$r^2=(0)^2+(-3)^2$$
$$=9$$
$$\therefore r=3 \text{ and the equation is } x^2+y^2=9$$

4. $$x^2+y^2=\left(2\frac{1}{2}\right)^2$$
$$\therefore x^2+y^2=\left(\frac{5}{2}\right)^2$$
$$\therefore x^2+y^2=\frac{25}{4} \quad \text{or } 4x^2+4y^2=25$$

Activity 2

1a



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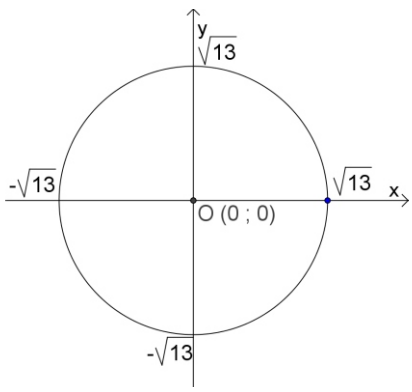
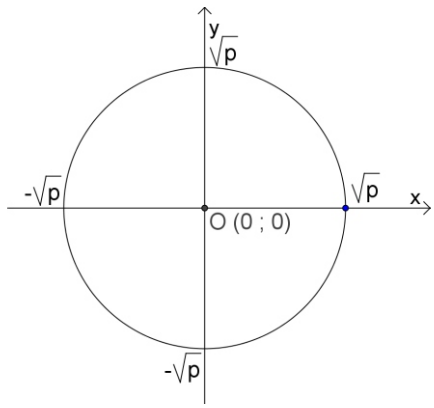


c



2. $$r^2=32$$
$$\therefore r=\sqrt{32}$$
$$=4\sqrt{2}$$

3. $$x=q,\quad y=4$$
$$\therefore q^2=28-(4)^2$$
$$=28-16$$
$$=12$$
$$\therefore q=\pm\sqrt{12}$$
$$=\pm\ 2\sqrt{3}$$

Activity 3

1. a) Centre $(-2;4)$, radius 4

   b) Centre $(0;3)$, radius $\sqrt{15}$

   c) Centre $(p;-4)$, radius $\sqrt{q}$

2. $(x-3)^2+(y-(-4))^2=r^2$

$(x-3)^2+(y+4)^2=r^2$

$\therefore(-2-3)^2+(3+4)^2=r^2$

$\therefore 25+49=r^2$

$\therefore$ the equation is $(x-3)^2+(y+4)^2=74$

3. $(x-2)^2+(y+2)^2=r^2$

$\therefore(0-2)^2+(y+2)^2=r^2$

$\therefore r^2=8$

$\therefore$ the equation is $(x-2)^2+(y+2)^2=8$

Activity 4

1. $x^2+2x+\left(\frac{1}{2}\times 2\right)^2+y^2-6y+\left(\frac{1}{2}\times 6\right)^2=6+1+9$ Completing the squares

$\therefore x^2+2x+1+y^2-6y+9=16$

$\therefore(x+1)^2+(y-3)^2=16$ Factorizing the trinomials

$\therefore$ the centre is $(-1;3)$ and the radius is 4.

2. $x^2+4x+(2)^2+y^2+2y+(1)^2=\frac{2}{3}+4+1$

$(x+2)^2+(y+1)^2=\frac{17}{3}$

$\therefore$ the centre is $(-2;-1)$ and the radius is $\sqrt{\frac{17}{3}}$.

3. $2x^2+2y^2+4y-6x-18=0$ We have to ensure that the coefficients of x and y are 1, so we first need to divide through by 2

$x^2+y^2+2y-3x-9=0$

$x^2-3x+\left(\frac{3}{2}\right)^2+y^2+2y+(1)^2=9+\left(\frac{3}{2}\right)^2+(1)^2$ Completing the 2 perfect squares

$\therefore\left(x-\frac{3}{2}\right)^2+(y+1)^2=\frac{49}{4}$ Factorize the two trinomials

$\therefore$ the centre is $\left(\frac{3}{2};-1\right)$ and the radius is $\frac{7}{2}$.

Activity 5

1. Centre of the circle is $(3;-2)$

$\therefore m_r=\frac{-2-(-3)}{3-1}=\frac{1}{2}$

$\therefore m_t=-2$

$\therefore y=-2x+c$

Using the point $(1;-3)$ we get $-3=-2(1)+c$

$c=-1$

The equation of the tangent is $y=-2x-1$

2. To find the centre of this circle, we'll complete the two square trinomials on the left hand side

$x^2+4x+(2)^2+y^2+2y+(1)^2=5+4+1$ — Halve the coefficient of x and square it

$\therefore (x+2)^2+(y+1)^2=10$ — Halve the coefficient of y and square it. Then balance the equation on the right

The centre of the circle is $\therefore (-2;-1)$

Gradient of the radius: $m_r=\dfrac{-2-(-1)}{1-(-2)}=-\dfrac{1}{3}$

Gradient of the tangent: $m_t=3$

Using the point $(1;-2)$ $\quad -2=3(1)+c$

$\therefore c=-5$

$\therefore$ Equation of the tangent is $y=3x-5$

3. $x^2+4x+y^2-2y=5$ — Reduce the coefficients of both x and y to 1

$x^2+4x+(2)^2+y^2-2y+(1)^2=5+4+1$ — Complete the two square trinomials

$(x+2)^2+(y-1)^2=10$ — Factorizing

$\therefore$ centre is $(-2;1)$

$m_r=\dfrac{4-1}{-1-(-2)}=3$

$\therefore m_t=-\dfrac{1}{3}$

Using point $(-1;4)$ $\quad 4=-\dfrac{1}{3}(-1)+c$

$\therefore c=\dfrac{11}{3}$

$\therefore$ Equation of the tangent is $y=-\dfrac{1}{3}x+\dfrac{11}{3}$ or $3y=-x+11$

Lesson 5

Activity 1

1. $3a^2+12a+9=0$

$\therefore 3(a^2+4a+3)=0$ (common factor of 3)

$\therefore a^2+4a+3=0$ (divide each side by 3)

$\therefore (a+3)(a+1)=0$ (factorise the trinomial)

$\therefore a+3=0$ or $a+1=0$

$\therefore a=-3$ or $a=-1$ (check your answers)

2. $4m^2-25=0$

$\therefore (2m-5)(2m+5)=0$ (difference of two squares)

$\therefore 2m-5=0$ or $2m+5=0$

$\therefore 2m=5$ or $2m=-5$

$\therefore m=\dfrac{5}{2}$ or $m=-\dfrac{5}{2}$

Remember:

All terms must go to the LHS and the RHS = 0

3. $p^2 = 4p$

$\therefore p^2 - 4p = 0$ (common factor of p)

$\therefore p(p-4)=0$

$\therefore p=0$ or $p-4=0$

$\therefore p=4$

4. $2(9x-4)=9x^2$

$\therefore 18x - 8 - 9x^2 = 0$

$\therefore -9x^2 + 18x - 8 = 0$

$\therefore 9x^2 - 18x + 8 = 0$ (multiply both sides with -1)

$\therefore (3x-2)(3x-4)=0$

$\therefore 3x-2=0$ or $3x-4=0$

$\therefore 3x = 2$ or $3x = 4$

$\therefore x = \frac{2}{3}$ or $x = \frac{4}{3}$

5. You can not solve the equation $x^2 + 4x - 3 = 0$ by factors.

Activity 2

$x^2 + x - 11 = 0$

$\therefore x^2 + x = 11$ (take variable terms to LHS, constant to RHS)

$\therefore x^2 + x + \left(\frac{1}{2}\right)^2 = 11 + \left(\frac{1}{2}\right)^2$ (complete the square by adding to both sides the square of half the coefficient of x)

$\therefore \left(x + \frac{1}{2}\right)^2 = \frac{45}{4}$ (simplify the LHS and RHS)

$\therefore x + \frac{1}{2} = \pm \frac{\sqrt{45}}{2}$ (take the square root on both sides)

$\therefore x = \frac{\sqrt{45}}{2} - \frac{1}{2}$ or $x = -\frac{\sqrt{45}}{2} - \frac{1}{2}$

$\therefore x = 2{,}85$ or $x = -3{,}85$

Activity 3

1. $a=1,\ b=-7$ and $c=12$

$$x_{1,2} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(12)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{49 - 48}}{2}$$

$$= \frac{7 \pm 1}{2}$$

$\therefore x = 4$ or $x = 3$

2. $a = 3$, $b = 0$ and $c = -8$

$$x_{1,2} = \frac{-(0) \pm \sqrt{(0)^2 - 4(3)(-8)}}{2(3)}$$

$$= \frac{\pm\sqrt{96}}{6}$$

$$\therefore x = \frac{\sqrt{96}}{6} \quad \text{or} \quad x = \frac{-\sqrt{96}}{6}$$

$\therefore x\ 1{,}63$ or $x = -1{,}63$ (correct to 2 decimal places)

3. $a = 2$, $b = 6$ and $c = 3$

$$x_{1,2} = \frac{-6 \pm \sqrt{36 - 24}}{4}$$

$$= \frac{-6 \pm \sqrt{12}}{4}$$

$$\therefore x_1 = \frac{-6 + \sqrt{12}}{6} \quad \text{or} \quad x_2\ \frac{-6 12\sqrt{\ }}{6}$$

$\therefore x_1 = -0{,}63$ or $x_2 = -2{,}37$ (correct to 2 decimal places)

Activity 4

1. $$\frac{x}{x-3} = \frac{6}{x-5}$$

$\therefore x(x-5) = 6(x-3)$ (multiply by the LCM which is $(x-3)(x-5)$)

$\therefore x^2 - 5x = 6x - 18$

$\therefore x^2 - 11x + 18 = 0$

$\therefore (x-9)(x-2) = 0$ (factorise as usual)

$\therefore x - 9 = 0$ or $x - 2 = 0$

$\therefore x = 9$ or $x = 2$

Be careful that you change all the signs inside the bracket when you multiply by a negative number as this is a common error.

2. $$\frac{4}{3x-2} - \frac{3}{2x-3} = \frac{1}{2x-1}$$

$\therefore 4(2x-3)(2x-1) - 3(3x-2)(2x-1) = (3x-2)(2x-3)$

$\therefore 4(4x^2 - 8x + 3) - 3(6x^2 - 7x + 2) = 6x^2 - 13x + 6$

$\therefore 16x^2 - 32x + 12 - 18x^2 + 21x - 6 = 6x^2 - 13x + 6$

$\therefore -8x^2 + 2x = 0$

$\therefore -2x(4x - 1) = 0$

$\therefore -2x = 0$ or $4x - 1 = 0$

$\therefore x = 0$ or $x = \frac{1}{4}$

3. $\dfrac{x}{3x-6}-\dfrac{2}{2-x}=2x$

$\therefore \dfrac{x}{3(x-2)}+\dfrac{2}{x-2}=2x$ (factorize and switch around the denominator on the LHS)

$\therefore x+2(3)=2x(3(x-2))$ (multiply through by LCM which is $3(x-2)$)

$\therefore x+6=6x(x-2)$

$\therefore x+6-6x^2+12x=0$

$\therefore -6x^2+13x+6=0$

$\therefore 6x^2-13x-6=0$

We cannot find the factors for this trinomial so we use the formula

$a=6$, $b=-13$ and $c=-6$

$\therefore x=\dfrac{-(-13)\pm\sqrt{(-13)^2-4(6)(-6)}}{2(6)}$

$\therefore x=\dfrac{13\pm\sqrt{169+144}}{12}$

$\therefore x=\dfrac{13\pm\sqrt{313}}{12}$

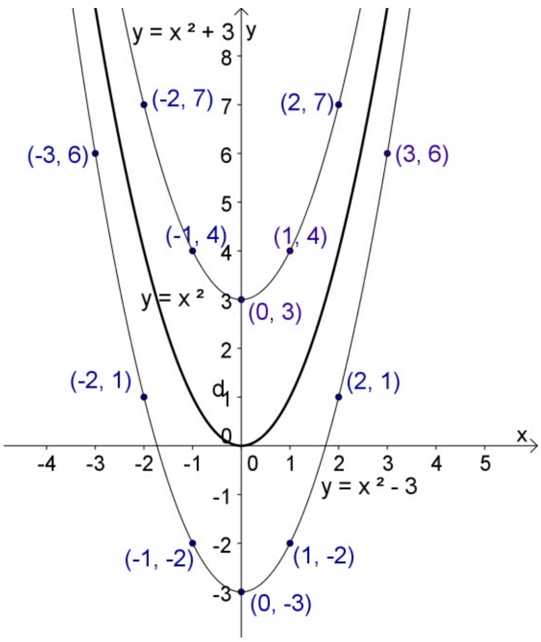
$\therefore x=\dfrac{13+\sqrt{313}}{12}$ or $x=\dfrac{13-\sqrt{313}}{12}$

$\therefore x=2{,}56$ or $x=-0{,}39$

Lesson 6

Activity 1

To help you see better the movement of the graphs we will draw these two graphs on the same axes. We will first draw the graph of $y=x^2$. We will then draw the two graphs, showing them as shifts from the graph of $y=x^2$. In mathematics, this shift is sometimes called a *translation*.

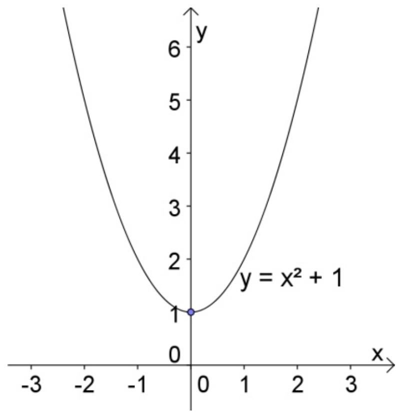


The graph of the form $y = x^2 + c$, is the same as the graph of $y = x^2$ that has shifted c units upwards along the y –axis, if $c > 0$. The graph will shift c units downwards if c is < 0. Any graph of the form of $y = x^2 + c$ is a graph of the same shape as $y = x^2$ shifted upwards or downwards along the y –axis, depending on the sign of the c.

Activity 2

These are all graphs of $y = x^2$ that have shifted upwards or downwards depending on the value added to or subtracted from x^2.

1.



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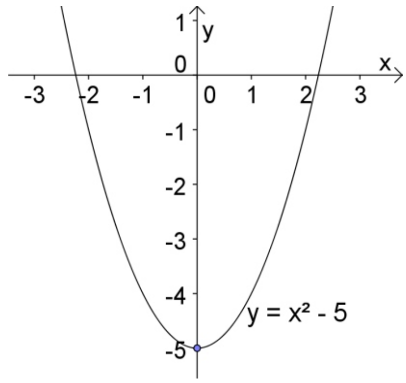

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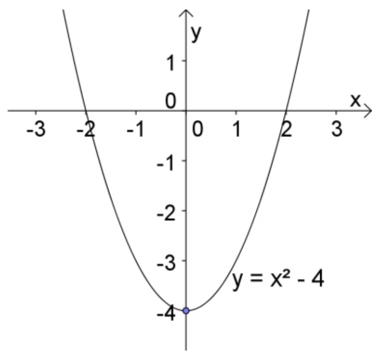

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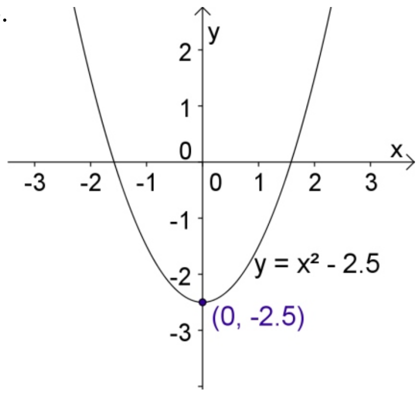
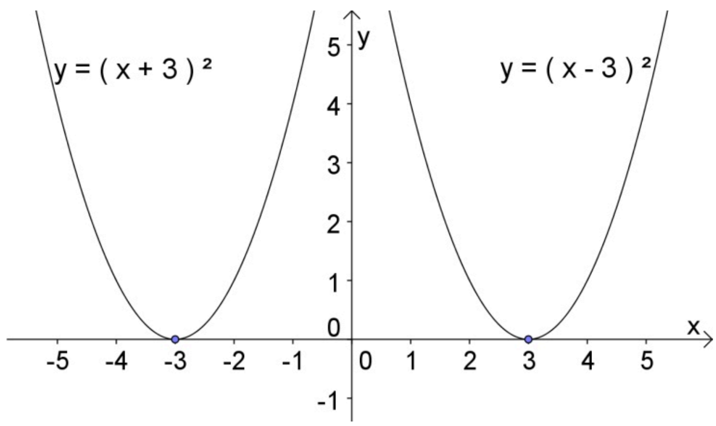


Activity 3

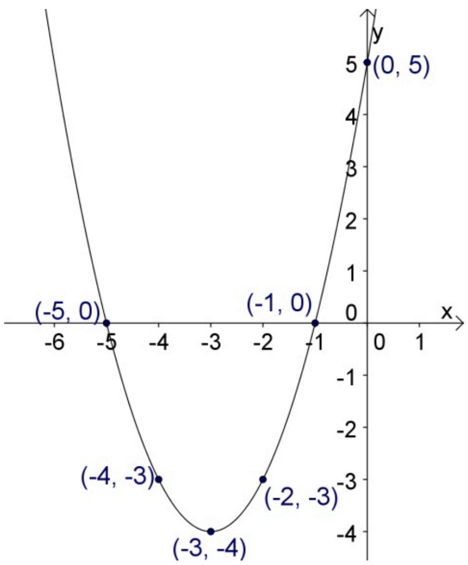


The graph of $y=(x-3)^2$ is the same as that of $y=x^2$ shifted 3 units to the right along the x –axis. The y –coordinates remain the same. The graph of $y=(x+3)^2$ is the same as that of $y=x^2$ shifted 3 units to the left along the x –axis. The y –coordinates remain the same.

The graph of $y=(x-p)^2$ is the same as the graph of $y=x^2$ that has shifted p units to the right along the x –axis if $p>0$. It will shift p units to the left if p is < 0.

Activity 4

x	- 4	–3	-2	–1	0	1
$x+3$	-1	0	1	2	3	4
$(x+3)^2$	1	0	1	4	9	16
$(x+3)^2-4$	-3	- 4	- 3	0	5	12



$y=(x+3)^2-4$ is a function of $y=x^2$ that has shifted 3 units to the left along the x –axis and 4 units downwards along the y –axis. Did you get the same thing?

The graph of $y=(x-p)^2+q$ is a graph of $y=x^2$ that has made both shifts of the graphs $y=(x-p)^2$ and $y=x^2+q$

The graph of $y=(x-p)^2+q$ will shift h units to the left or right and q units up or down depending on the signs of p and q.

We can predict the shift of the graph just by looking at the formula of the function.

Activity 5

1. $y=x^2+8x+15$

$$\therefore y=x^2+8x+16-16+15$$
$$=(x+4)^2-1$$

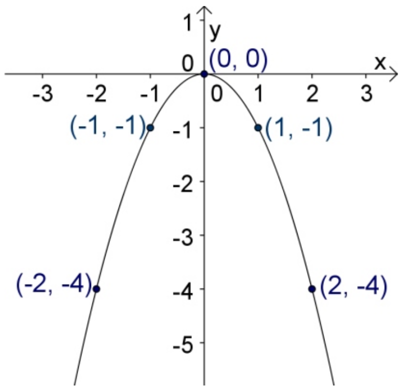
2. $y=x^2-x+3$

$$\therefore y=x^2-x+\frac{1}{4}-\frac{1}{4}+3$$
$$=\left(x-\frac{1}{2}\right)^2+2\frac{3}{4}$$

Activity 6

After drawing the graphs we can see that all the graphs look very similar to the graphs when $a > 0$.
The only difference is that they appear to be upside down. We call these minimum point parabolas. Did you notice the same thing? Here is what they look like:

1.



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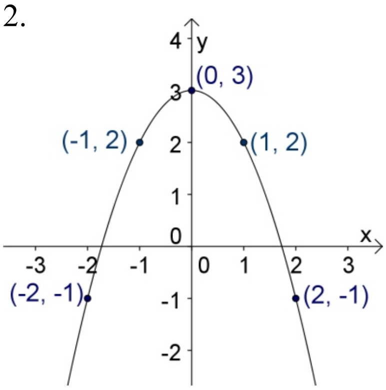

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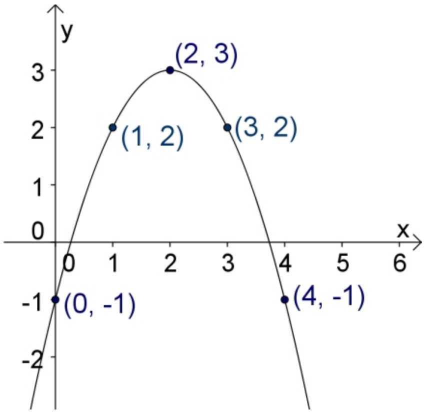


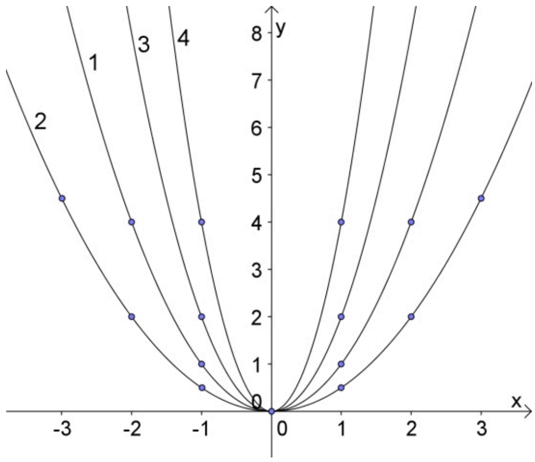
Activity 7

Let us make a table of co-ordinates like this:

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
$\frac{1}{2}x^2$	4,5	2	0,5	0	0,5	2	4,5
$2x^2$	18	8	2	0	2	8	18
$4x^2$	36	16	4	0	4	16	36

The drawings show the changes that we should expect if we have $y = ax^2$, where a is either greater or less than 1.

When $0 < a \leq 1$, the graph of $y = ax^2$ flattens out. It becomes a wider 'cup'. When $a > 1$, the graph of $y = ax^2$ gets narrower. The bigger a is, the steeper the graph.

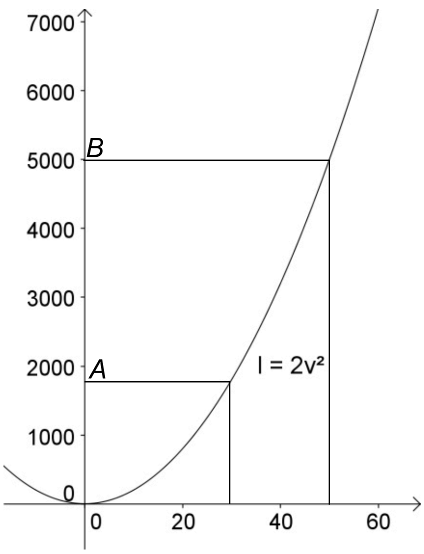


Activity 8

The drawing of the graph may seem to be a waste of time in this question. In the case of routine work, it is useful to have a chart or graph from which workers will just read the answers without doing calculations.

The table of points of the graph of $I = 2v^2$ is as follows:

v	0	1	20	30	40	50	60
I	0	200	800	1 800	3 200	5 000	7 200



Impact, I, is on the vertical axis, and the speed, v is on the horizontal axis.

When $v = 30$, $I = 1\ 800$ (at A) and when $v = 50$, $I = 5\ 000$ (at B).

If we know the amount of damage caused, and the force that this car applied to another object on collision, we can use this graph to find the speed of the car at the time of the collision.

If we are told that the impact was 6 000, we can read from the graph that the speed of the car must have been a little bit less than 55 km/h.

Lesson 7

Activity 1

1. a

2. c

3. b

4. e

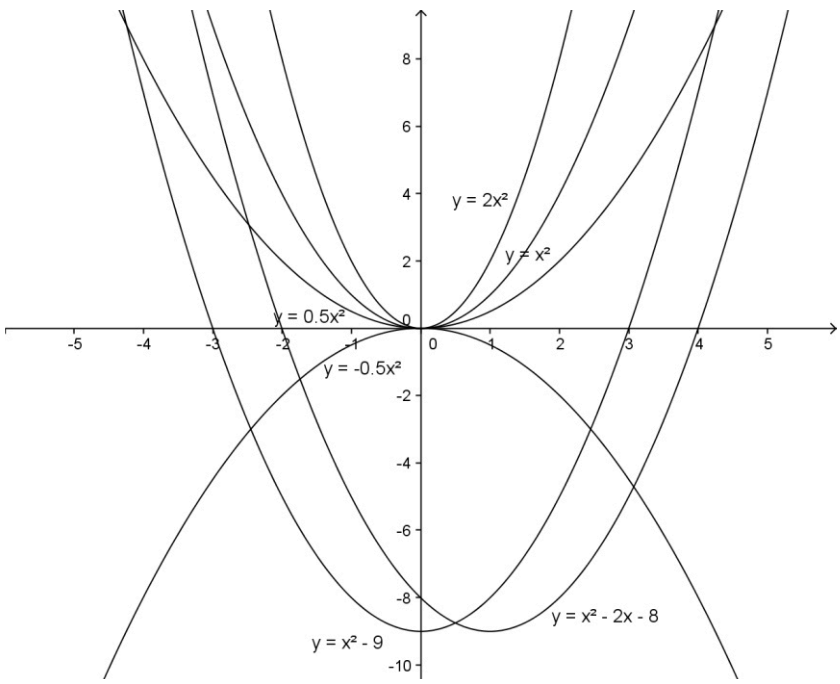
5. d

Activity 2

1. Example of the table method for $y = x^2$.

x	-3	-2	-1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9

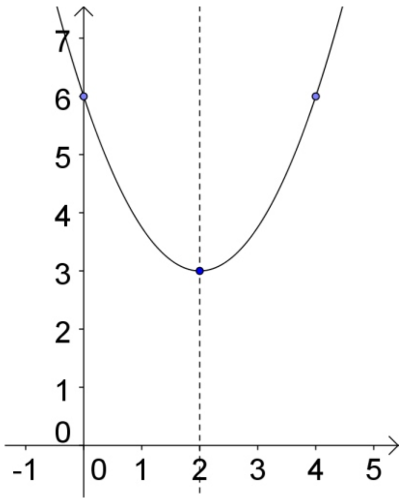
Plot these points on the Cartesian plane to see the quadratic function of $y = x^2$. Use the same method to find the curves of the other functions.



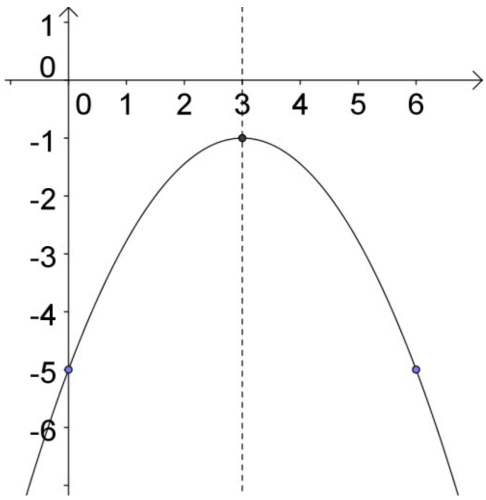
2. a) changing the graph from smiling to frowning
   b) moves the graph sideways
   c) moves the graph up or down

Activity 3

1.



2.



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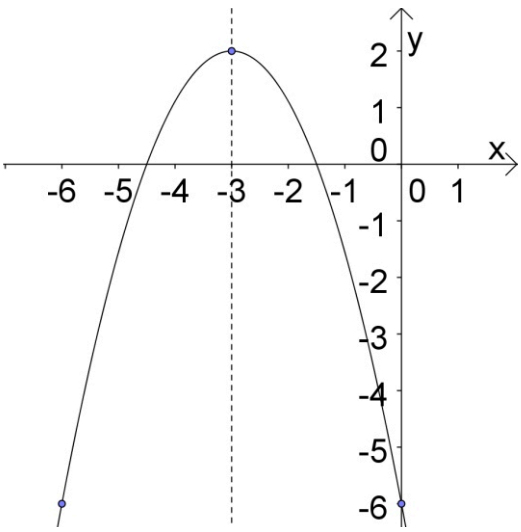

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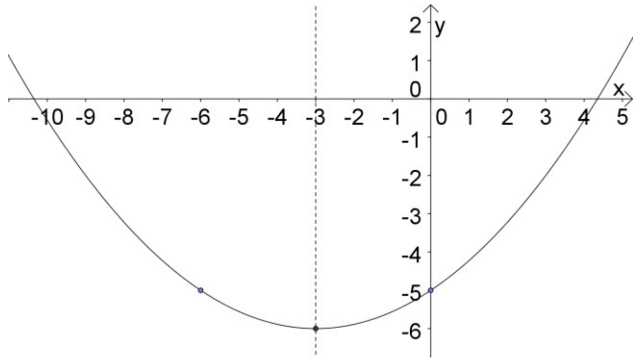

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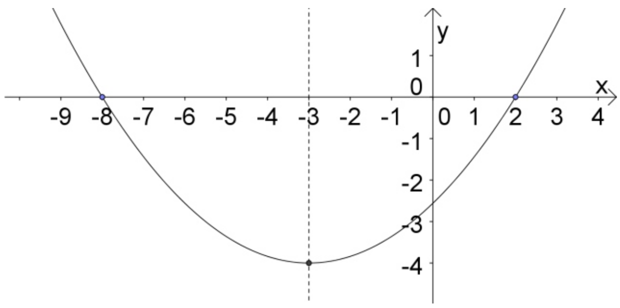

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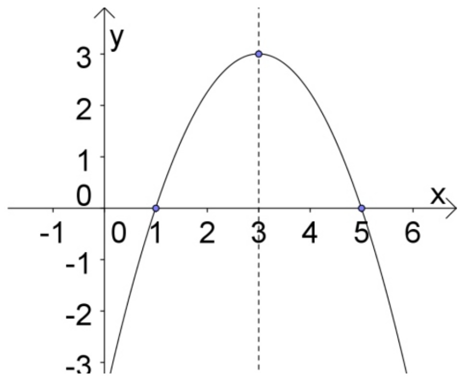


Activity 4

1. $y = 2x^2 + 4x + 6$ $\quad \therefore a = 2, b = 4$ and $c = 6$

Axis of symmetry: $x = \dfrac{-b}{2a}$

$$\therefore x \ \frac{-4}{2(2)}$$

$$= -1$$

2. $y=-x^2+4x-4$ $\quad\therefore a=-1,\ b=4$ and $c=-4$

Axis of symmetry: $x=\dfrac{-b}{2a}$

$$\therefore x\ \frac{-4}{2(-1)}$$

$$=2$$

3. $y=x^2-4$ $\quad\therefore a=1,\ b=0$ and $c=-4$

Axis of symmetry: $x=\dfrac{-b}{2a}$

$$\therefore x\ \frac{-0}{2(1)}$$

$$=0$$

Activity 5

a) At the turning point $x=\dfrac{-b}{2a}$ so $x=\dfrac{-(-5)}{2(1)}=\dfrac{5}{2}$

y –value at turning point is $y=\left(\dfrac{5}{2}\right)^2-5\left(\dfrac{5}{2}\right)+2$

$$=\frac{25}{4}-\frac{25}{2}+2$$

$$=\frac{25-50+8}{4}$$

$\therefore TP\left(\dfrac{5}{2};-\dfrac{17}{4}\right)$ $\qquad =-\dfrac{17}{4}$

b) At the turning point $x=\dfrac{-b}{2a}$ so $x=\dfrac{-(3)}{2(2)}=-\dfrac{3}{4}$

y –value at turning point is $y=2\left(-\dfrac{3}{4}\right)^2+3\left(-\dfrac{3}{4}\right)-1$

$$=\frac{18}{16}-\frac{9}{4}-1$$

$$=\frac{18-36-16}{16}$$

$\therefore TP\left(-\dfrac{3}{4};-\dfrac{17}{8}\right)$ $\qquad =-\dfrac{34}{16}\quad=-\dfrac{17}{8}$

Activity 7

I suggest you look at the 6 points under 'Sketching the parabola'.

a) $y = x^2 - x - 2$

1. $a > 0 \ \therefore \cup$ shaped
2. $c = -2 \therefore$ the y - intercept $= -2$
3. $x^2 - x - 2 = 0$

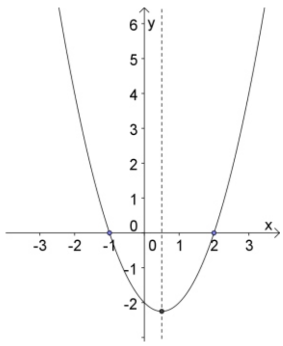
   $\therefore (x-2)(x+1) = 0$

   $\therefore x = 2 \quad \text{or} \quad x = -1$
4. A/S $\quad x = \frac{-b}{2a} = \frac{-(-1)}{2(1)} = \frac{1}{2}$
5. y –value at turning point $\quad y = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2$

   $\therefore y = \frac{11}{42} - - - 2$

   $= -2\frac{1}{4}$

   $\therefore TP\left(\frac{1}{2}; -2\frac{1}{4}\right)$



b) $y = -x^2 - 2x + 3$

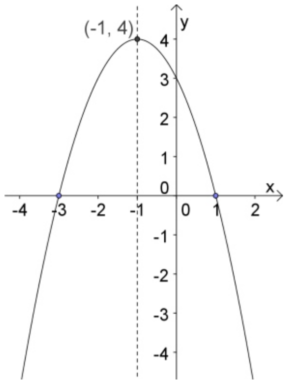
1. $a < 0 \ \therefore \cap$ shaped
2. $c = 3 \ \therefore$ the y - intercept $= 3$
3. $-x^2 - 2x + 3 = 0$

   $x^2 + 2x - 3 = 0$

   $\therefore (x+3)(x-1) = 0$

   $\therefore x = -3 \quad or \quad x = 1$
4. A/S $\quad x = \frac{-b}{2a} = \frac{-(-2)}{2(-1)} = -1$
5. y –value at turning point $\quad y = -(-1)^2 - 2(-1) + 3$

   $\therefore y = -1 + 2 + 3$

   $\therefore TP(-1;4) \qquad = 4$
6. 

c) $y = x^2 + 6x - 5$

1. $a > 0 \therefore \cup$ shaped
2. $c = -5 \therefore$ the y-intercept $= -5$
3. $x^2 + 6x - 5 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(-5)}}{2(1)}$$

$$\therefore x = \frac{-6 \pm \sqrt{56}}{2}$$

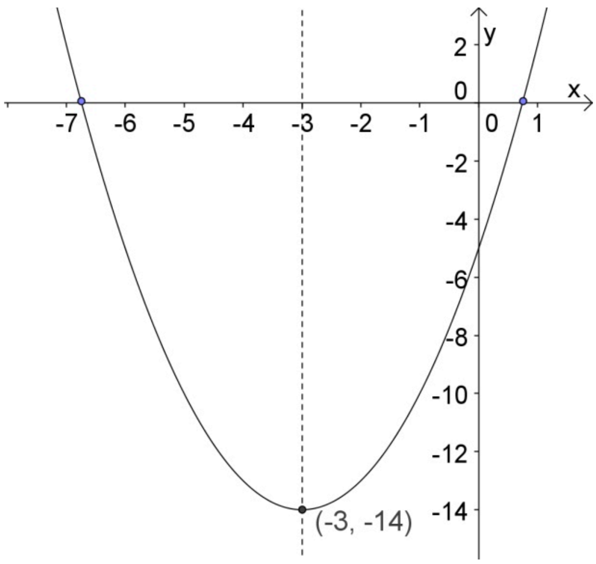
$\therefore x = 0.74$ or $x = -6.74$

4. A/S $x = \frac{-b}{2a} = \frac{-(6)}{2(1)} = -3$

5. y–value at turning point $y = (-3)^2 + 6(-3) - 5$

$\therefore y = 9 - 18 - 5$

$\therefore TP(-3; -14)$ $= -14$

6. 

d) $y = x^2 - 6x + 9$

1. $a > 0 \therefore \cup$ shaped
2. $c = 9 \therefore$ the y- intercept $= 9$
3. $x^2 - 6x + 9 = 0$

$\therefore (x - 3)(x - 3) = 0$

$\therefore x = 3$

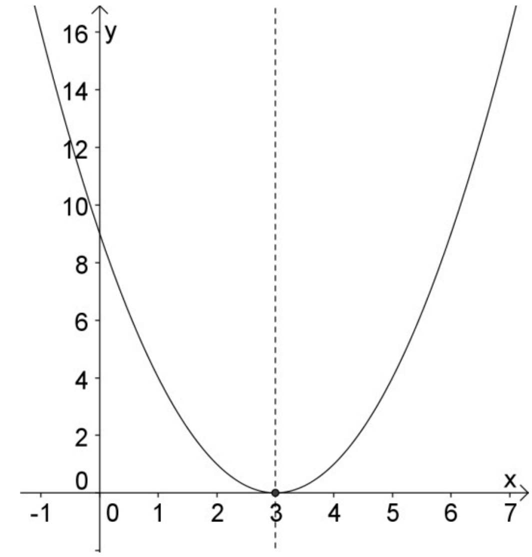
4. A/S $x = \frac{-b}{2a} = \frac{-(-6)}{2(1)} = 3$

5. y–value at turning point $y = (3)^2 - 6(3) + 9$

$\therefore y = 9 - 18 + 9$

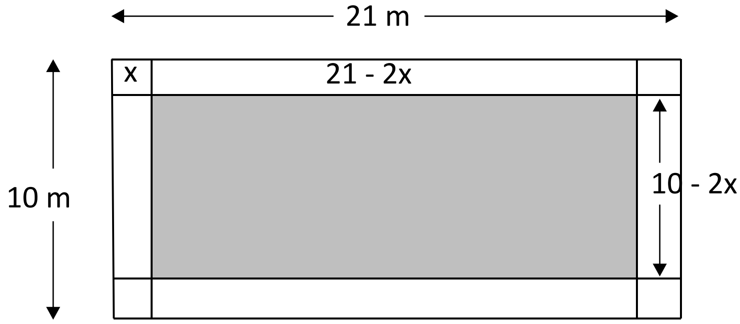
$\therefore TP(3; 0)$ $= 0$

6.



Activity 8

Let the thickness of the walls be x m. A diagram of the house will be useful at this stage.



Then, the length inside the house is $21-2x$ and the width is $10-2x$.

Inside Area = $(21-2x)(10-2x)=180$

$$\therefore 210-62x+4x^2=180$$

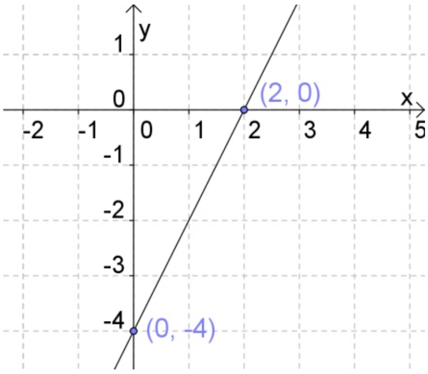
$$\therefore 4x^2-62x+30=0$$

Solving for x in this equation is solving for the thickness of the wall. I will leave that one with you.

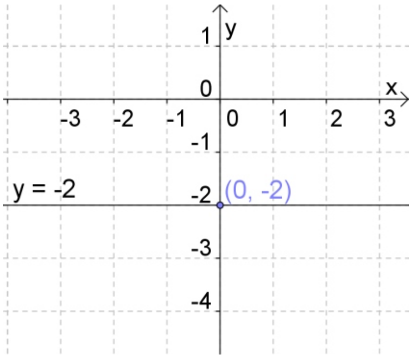
Feedback to Self-Check Exercises

Lesson 2

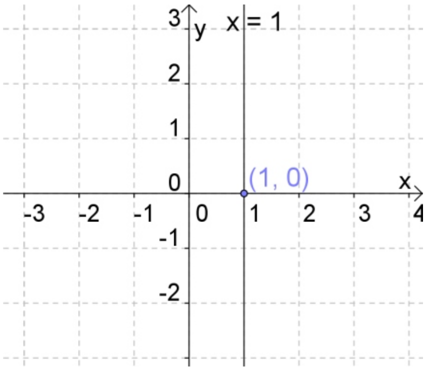
1a)



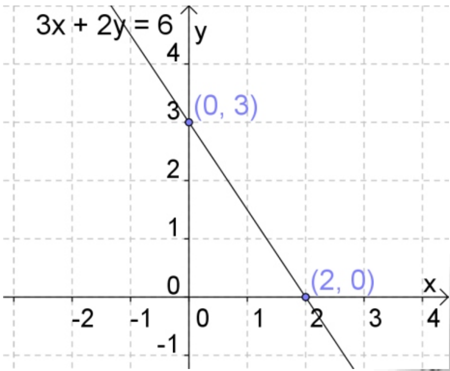
b)



c)



d)



2a) parallel lines have equal gradients so $p = 3$.

b) perpendicular lines have gradients that when multiplied give -1 so $p = -\frac{1}{3}$

c) $p = -1$

d) $p = 1$

3a) $(3; -2)$ parallel to $y = (3x + 1)$

Lines are parallel and therefore have the same gradient $\therefore m = 3$ with $x_1 = 3$ and $y_1 = -2$

$y - y_1 = m(x - x_1)$ (substitute given values)

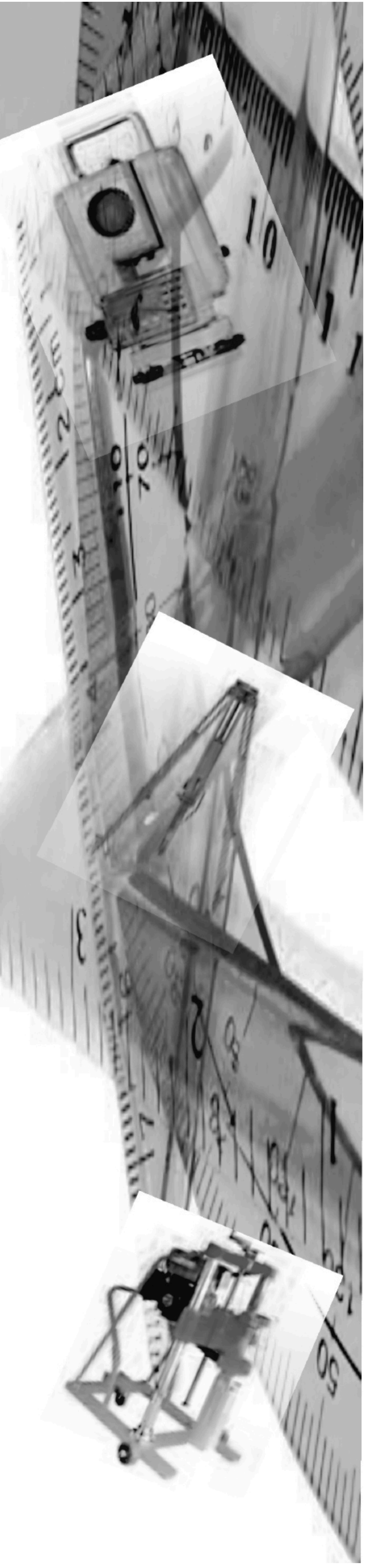
$y - (-2) = 3(x - 3)$ (expand the right hand side)

$y + 2 = 3x - 9$

$y = 3x - 11$

b) $(3;-2)$ perpendicular to $y = 3x + 1$

Lines are perpendicular $\therefore m = -\frac{1}{3}$ with $x_1 = 3$ and $y_1 = -2$



$$y - y_1 = m(x - x_1)$$
$$y - (-2) = -\frac{1}{3}(x - 3)$$
$$y + 2 = -\frac{1}{3}x + 1$$
$$y = -\frac{1}{3}x - 1 \quad \text{or } 3y = -x - 3$$

4a) $2x - y = 0 \qquad (2;4)$

$$LHS = 2(2) - 4$$
$$= 0$$

The point (2;4) does lie on the line $2x - y = 0$

b) $A(4;5);\ B(2;-1);\ C(0;-3)$

$$m_{AB} = \frac{-1-5}{2-4} = \frac{-6}{-2} = 3$$
$$m_{AB} = \frac{-3+1}{0-2} = \frac{-2}{-2} = 1$$

$\therefore$ the points are **not** collinear

Lesson 3

1a) $m_{AB} = \frac{1}{2} \qquad m_{CD} = \frac{1}{2} \qquad \therefore \text{AB} \parallel \text{CD}$

b) $m_{AB} = \frac{3}{5} \qquad m_{CD} = \frac{-7}{-7} = 1$ $\therefore$ lines are **not** parallel or perpendicular

c) $m_{AB} = \frac{3}{4} \qquad m_{CD} = \frac{4}{3}$ $\therefore$ lines are **not** parallel or perpendicular

2. $A(-2;3); B(-3;2)$

$$AB = \sqrt{-3-(-2)^2 + (2-3)^2}$$
$$= \sqrt{(-1)^2 + (-1)^2}$$
$$= \sqrt{2}$$

3a) $A(-3;7); B(1;-2)$

$$m = \frac{-2-7}{1-(-3)} = -\frac{9}{4}$$
$$\therefore y - 7 = -\frac{9}{4}(x - (-3))$$
$$\therefore y - 7 = -\frac{9}{4}x - \frac{27}{4}$$
$$\therefore y = -\frac{9}{4}x + \frac{1}{4} \quad \text{or } 4y = -9x + 1$$

The lines are parallel therefore the gradients are the same

b) Parallel to $y = -\frac{9}{4}x + \frac{1}{4} \therefore m = -\frac{9}{4}$ with $x_1 = -1$ and $y_1 = 2$

$\therefore y - 2 = -\frac{9}{4}(x - (-1))$

$\therefore y - 2 = -\frac{9}{4}x - \frac{9}{4}$

$\therefore y = -\frac{9}{4}x - \frac{1}{4}$ or $4y = -9x - 1$

The lines are perpendicular therefore the gradients are inverse and opposite in sign

c) Perpendicular to $y = -\frac{9}{4}x + \frac{1}{4} \therefore m = \frac{4}{9}$ with $x_1 = -1$ and $y_1 = 2$

$\therefore y - 2 = \frac{4}{9}(x - (-1))$

$\therefore y - 2 = \frac{4}{9}x + \frac{4}{9}$

$\therefore y = -\frac{9}{4}x + 2\frac{4}{9}$ or $36y = -81x + 88$

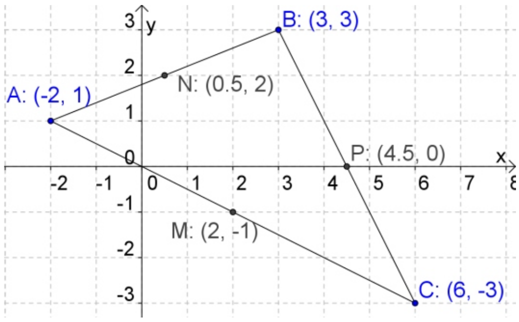
4a) $A(-2; 1)$;

$M = \left(\frac{-2+6}{2}; \frac{1-3}{2}\right)$

$= (2; -1)$

b) $N\left(\frac{1}{2}; 2\right)$

c) $P\left(4\frac{1}{2}; 0\right)$



Lesson 4

1a) Centre $(0; 0)$ Radius $\sqrt{17}$

b) Centre $(2; 3)$ Radius 3

c) $x^2 + 4x + (2)^2 + y^2 + 4y + (2)^2 = 18 + 4 + 4$

$\therefore (x+2)^2 + (y+2)^2 = 26$

Centre $(-2; -2)$ Radius $\sqrt{26}$

d) $x^2 + y^2 + 2y + (1)^2 = 6 + 1$

$x^2 + (y+1)^2 = 7$

Centre $(0; -1)$ Radius $\sqrt{7}$

2. If AB is the diameter, then the centre of the circle must be half way across AB, in other words, the midpoint of AB

$$C=\left(\frac{-1+5}{2};\frac{1+(-3)}{2}\right)=(2;-1)$$

$\therefore$ the equation is $(x-2)^2+(y+1)^2=r^2$

To find r, we substitute one of the points on the circle's edge. Let's use (–1; 1)

$$(-1-2)^2+(1+1)^2=r^2$$

$$\therefore r^2=9+4$$

$\therefore$ the equation is $(x-2)^2+(y+1)^2=13$

3. $x^2-4x+(2)^2+y^2=0+4$

$$\therefore (x-2)^2+y^2=4$$

$\therefore$ centre of circle is (2; 0)

$m_r=\dfrac{2-0}{2-2}$ which is undefined

An undefined gradient means that the line is a vertical line, parallel to the y–axis

$\therefore$ the tangent must be a horizontal line parallel to the x–axis

$\therefore\ m_t=0$

The equation of the tangent becomes $y=0x+c$

Substitute the point of contact (2; 2)

$$2=0(2)+c$$

$$\therefore c=2$$

The equation of the tangent is $y=2$

4a) $m_{BC}=\dfrac{1-2}{2-(-1)}=-\dfrac{1}{3}$

$$\therefore y=-\frac{1}{3}x+c$$

$$1=-\frac{1}{3}(2)+c$$

$$\therefore c=\frac{5}{3}$$

$$\therefore BC:\ y=-\frac{1}{3}x+\frac{5}{3}$$

b) $m_{BD}=3$ $(m_{BC}\text{ x }m_{BD}=-1)$

$$\therefore y=3x+c$$

$$1=3(2)+c$$

$$\therefore c=-5$$

$$\therefore BD:\ y=3x-5$$

c) $D(0;-5)$

$\therefore$ the centre of the circle is $\left(\frac{2+0}{2};\ \frac{1+(-5)}{2}\right) = (1;-2)$

$(x-1)^2 + (y+2)^2 = r^2$

$\therefore (2-1)^2 + (1+2)^2 = r^2$

$\therefore r^2 = 10$

Equation of the circle is $(x-1)^2 + (y+2)^2 = 10$

Lesson 5

1. $x^2 - 5x - 6 = 0$

$\therefore (x-6)(x+1) = 0$

$\therefore x = 6 \quad or \quad x = -1$

$\therefore$ b) is the solution

2a) $x^2 - 4x + 4 = 0$

$\therefore (x-2)(x-2) = 0$

$\therefore x = 2$

b) $x^2 + 14x = 32$

$\therefore x^2 + 14x + \left(\frac{14}{2}\right)^2 = 32 + \left(\frac{14}{2}\right)^2$

$\therefore (x+7)^2 = 81$

$\therefore x + 7 = \pm 9$

$\therefore x + 7 = 9 \quad or \quad x + 7 = -9$

$\therefore x = 2 \quad or \quad x = -16$

c) $3x^2 - 4x - 2 = 0$ with $a = 3; b = -4$ and $c = -2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{4 \pm \sqrt{40}}{6}$$

$$\therefore x = \frac{4 + \sqrt{40}}{6} \quad or \quad x = \frac{4 - \sqrt{40}}{6}$$

$\therefore x = 1,72 \quad or \quad x = -0,39$ (correct to 2 decimal places)

d) $\frac{x+2}{x-2}=1-\frac{x+8}{x+2}$

$\therefore (x+2)(x+2)=(x+2)(x-2)-(x+8)(x-2)$ (multiply both sides by the LCM)

$\therefore x^2+4x+4=x^2-4-(x^2+6x-16)$

$\therefore x^2+4x+4=x^2-4-x^2-6x+16$

$\therefore x^2+10x-8=0$

$$x=\frac{-10\pm\sqrt{(10)^2-4(1)(-8)}}{2(1)}$$

$$=\frac{-10\pm\sqrt{132}}{2}$$

$\therefore x=\frac{10+\sqrt{132}}{2}$ *or* $x=\frac{-10-\sqrt{132}}{2}$

$\therefore x=0,74$ *or* $x=-10,74$ (correct to 2 decimal places)

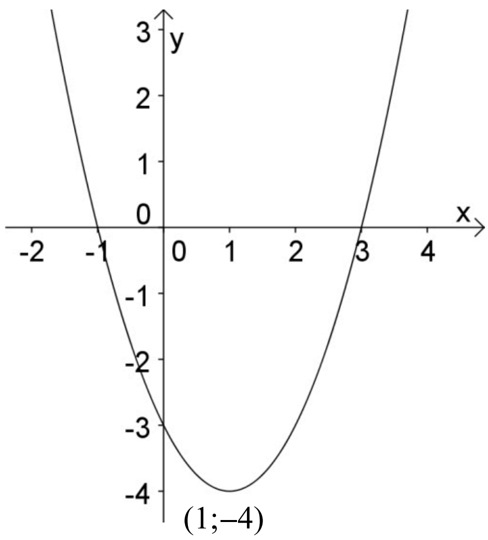
Lesson 6

1a) $y=x^2-2$

1b) $y=-(x-2)^2$

1c) $y=-(x-2)^2+2$

2.



Lesson 7

2. $a>0$ then the curve $\cup$
$a<0$ then the curve $\cap$
$a=0$ then the curve is a straight line

3a) Axis of Symmetry: $x=\frac{-b}{2a}$

If $y=-x^2+2x+3$ then $a=-1$, $b=2$ and $c=3$

$\therefore x=\frac{-2}{2(-1)}=1$

$\therefore A/S:\ x=1$ This could also be read from the graph.

b) Read from the graph: (0; 3)

c) When $x = 0$: $-x^2 + 2x + 3 = -(0)^2 + 2(0) + 3$

$= 3$

When $x = 1$: $-x^2 + 2x + 3 = -(1)^2 + 2(1) + 3$

$= 4$

d) For maximum value, we need the turning point
To find turning point, we substitute the axis of symmetry ($x = 1$) into the equation
$\therefore \; y = -(1)^2 + 2(1) + 3 = 4$
$\therefore$ the maximum value is 4

e) Read from the graph: $x = -1$ and $x = 3$
To check algebraically, we make $y = 0$

$$-x^2 + 2x + 3 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$x = 3 \quad or \quad x = -1$$

f) From d, $TP\,(1; 4)$

4. $y = 2x^2 - 7x + 6$

Step 1: graph is $\cup$ since $a > 0$

Step 2: intercept on the y - axis is 6

Step 3: intercepts on the x - axis

$$2x^2 - 7x + 6 = 0$$
$$\therefore (2x - 3)(x - 2) = 0$$
$$\therefore x = \frac{3}{2} \quad or \quad x = 2$$

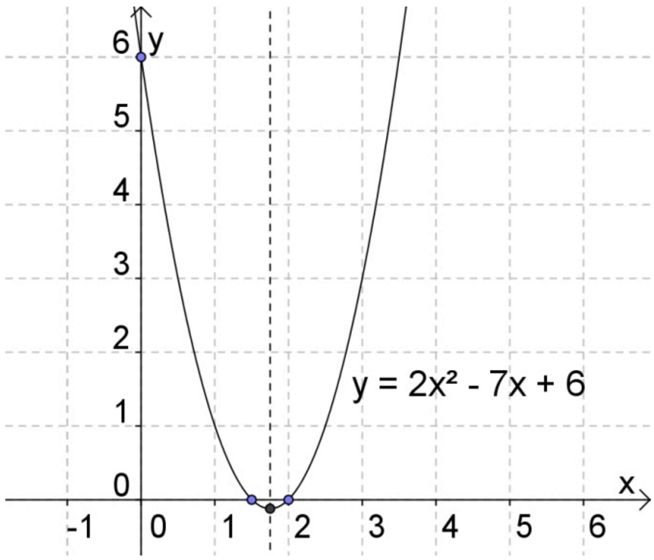
Step 4: axis of symmetry is $x = \frac{-(-7)}{2(2)} = \frac{7}{4}$

Step 5: turning point $y = 2\left(\frac{7}{4}\right)^2 - 7\left(\frac{7}{4}\right) + 6$

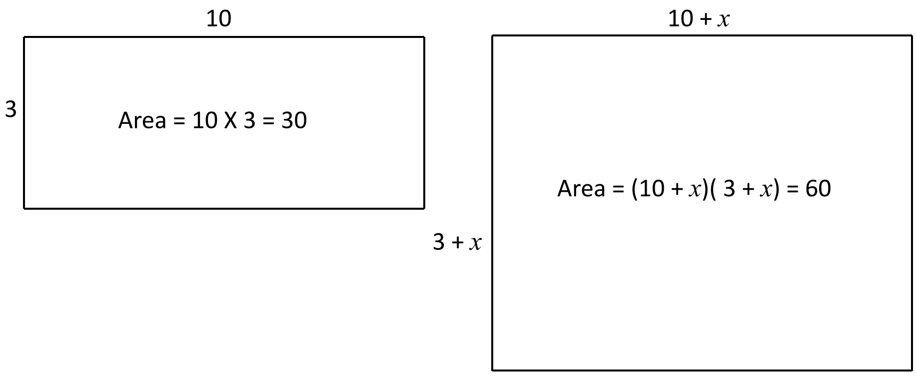
$$= \frac{49}{8} - \frac{49}{4} + 6$$
$$= \frac{1}{8}$$

$\therefore TP\; \frac{7}{4}; \frac{1}{8}$

Step 6:



5.



Area: 10 x 3 = 30 tiles

New dimensions: Across $10 + x$

Down $3 + x$

New Area: $(10 + x)(3 + x) = 60$ (Area doubles)

$\therefore\ 30 + 10x + 3x + x^2 = 60$

$\therefore\ x^2 + 13x - 30 = 0$

$\therefore\ (x + 15)(x - 2) = 0$

$\therefore\ x = -15 \quad or \quad x = 2$

Since -15 is impossible, $x = 2$

$\therefore$ the new number of tiles is 12 across and 5 down.