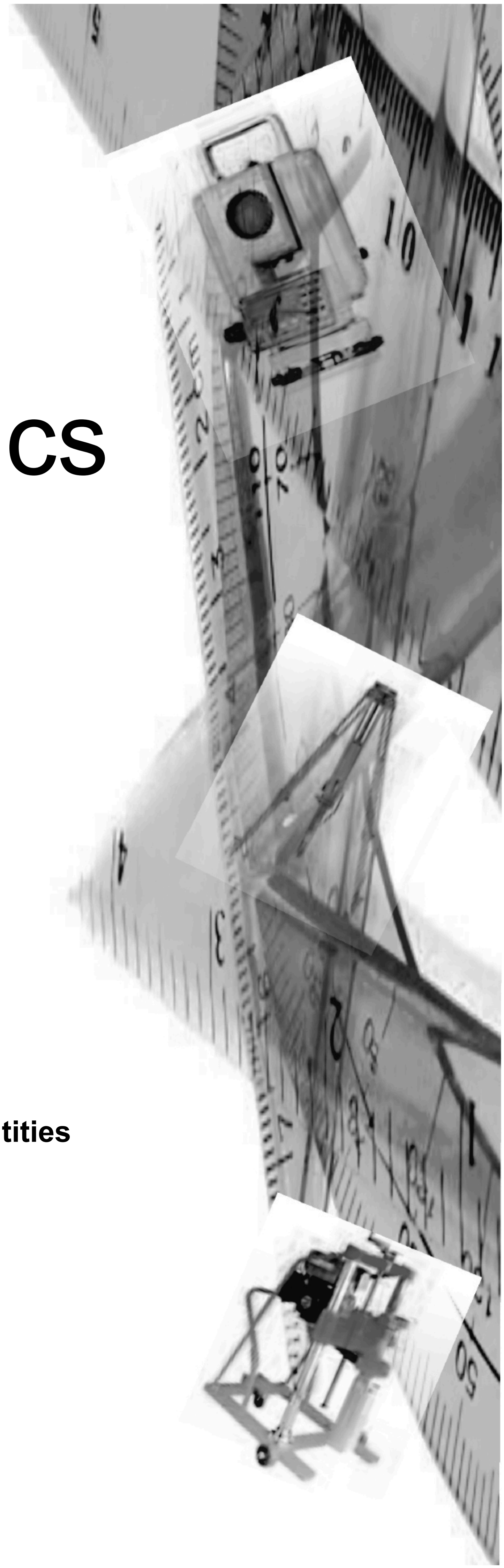

Mathematics

Unit 3

Quadratics, Trigonometric Identities & Triangles

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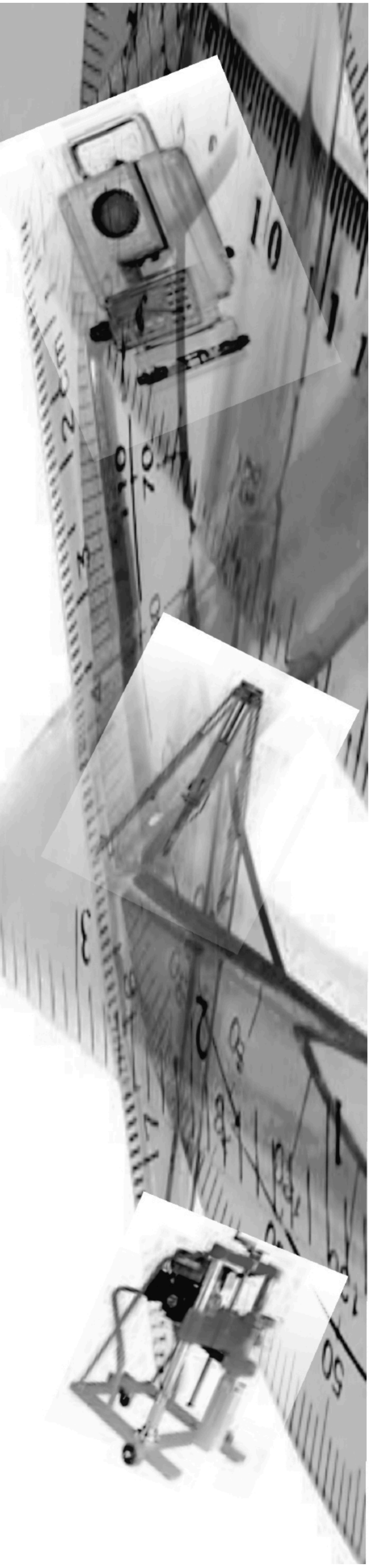
LESSON 1

More about Quadratics



About this lesson

You may think that all mathematical questions can be solved, if only a suitable method is used. However there are some problems that cannot be solved whatever method is used.

For example, you have been introduced to many methods of solving quadratic equations. What happens when the methods given do not give a solution? Are we able to do something about that?

There have been times when you were asked to solve a problem backwards. You have been finding roots of equations so far. There may be a situation when you have the roots, and you want the original equation. How do you go backwards? This lesson will answer these questions.

In this lesson you will:

- look at quadratic equations that have no solution
- use roots to find the original quadratic equation
- try to recognise some disguised quadratics
- solve some practical questions in the form of quadratics

When the quadratic formula does not work

Roots of quadratic equations can be rational, irrational or imaginary numbers. If you can't remember the difference between these different types of numbers, you will need to study this work again.

Remember that we are dealing with **real number functions**. This means that all our answers will be real numbers. We will not have imaginary numbers. We will not have answers that are the result of taking square roots of negative numbers. For us, something like this, $\sqrt{-9}$ is not a real number.

There are moments when we do things 'in the right way' but they don't work out. In mathematics these are exceptions to the rule or 'counter-examples'.

Look at the following example.

Example 1

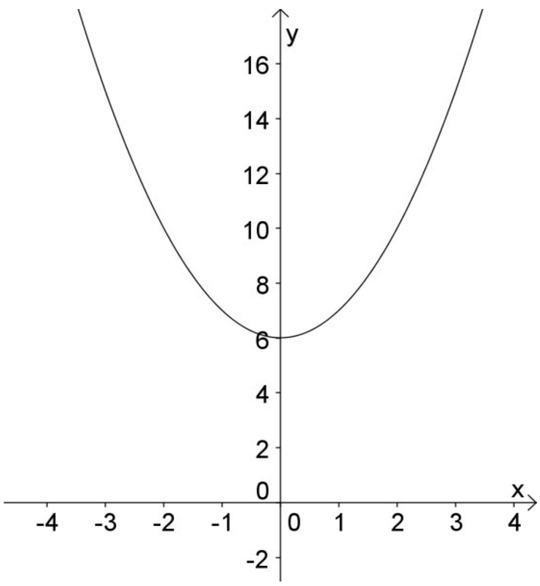
Try to use the quadratic formula to find where the graph of $f(x) = x^2 + 6$ cuts the x –axis.

Solution

In this function, $a = 1, b = 0$ and $c = 6$. Therefore,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{0 \pm \sqrt{0^2 - 4(1)(6)}}{2(1)} \\ &= \pm\frac{\sqrt{-24}}{2} \end{aligned}$$

The value under the square root sign is negative. The square root of a negative number is not a real number. When we draw the graph of this function, it does not cut the x –axis, as indicated in the sketch below.



This is a basic parabola that has shifted up 6 units. It won't cut the x –axis at all. So, non-real roots means that the parabola doesn't cut the x –axis.

Now try another example yourself.

ACTIVITY 1

Find roots of the following equation:

$f(x) = 3x^2 + 2x + 1$

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Let's see how we can work backwards to solve a problem.

Using roots to find the original equation

Sometimes you will be given the roots of an equation and you will be expected to find the original equation. Sometimes you will be told where the quadratic graph cuts the x –axis. You will be expected to find the original function. How can you do that?

To answer this question, let's look at an example.

Example 2
In the equation $x^2 - 7x + 12 = 0$, the roots are given as $x = 4$ or $x = 3$

Is it possible to get the original equation from the roots?

Solution
Consider what the last line of your answer would be if you were asked to solve the quadratic equation.

It would be	$x = 4$ or $x = 3$
The step before that would read	$x - 4 = 0$ or $x - 3 = 0$
Before that would be	$(x-4)(x-3) = 0$
Finally, the first step would be the equation itself	$x^2 - 7x + 12 = 0$

We use this process of thinking of the solution backwards to build the possible equation that could result in the given solutions.

Let us look at some more examples.

Example 3
Suppose we were asked to find a possible equation that had the roots 2 and $\frac{2}{3}$.

Solution

We step through the solution of a quadratic equation backwards to find the original equation.

The last step would be	$x = 2$ or $x = \frac{2}{3}$
The step before the last in getting the second root is:	$3x = 2$ (multiplying both sides by 3)
The step before that would read	$x - 2 = 0$ or $3x - 2 = 0$
Before that would be	$(x-2)(3x-2) = 0$
Finally, the first step would be the equation itself	$3x^2 - 8x + 4 = 0$

Example 4

Roots of an equation are $\frac{5}{6}$ and $-\frac{4}{3}$. Find the equation.

Solution

$x = \frac{5}{6}$ or $x = -\frac{4}{3}$

$\therefore 6x = 5$ or $3x = -4$

$\therefore 6x - 5 = 0$ or $3x + 4 = 0$

$\therefore (6x-5)(3x+4) = 0$

$\therefore 18x^2 + 9x - 20 = 0$

Do the following activity to check that you understand this.

ACTIVITY 2

Find the original equations that would result in the given roots:

a) 2;3
b) 2; –3
c) –2;3
d) –2; –3
e) 4;4
f) $-\frac{3}{2}$; $-\frac{1}{8}$

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Unusual Quadratics

Equations are often given in an unfamiliar form. At first glance they would not seem to be quadratic equations at all. Often, simplifying the given equations using ordinary algebra allows us to put them into standard form.

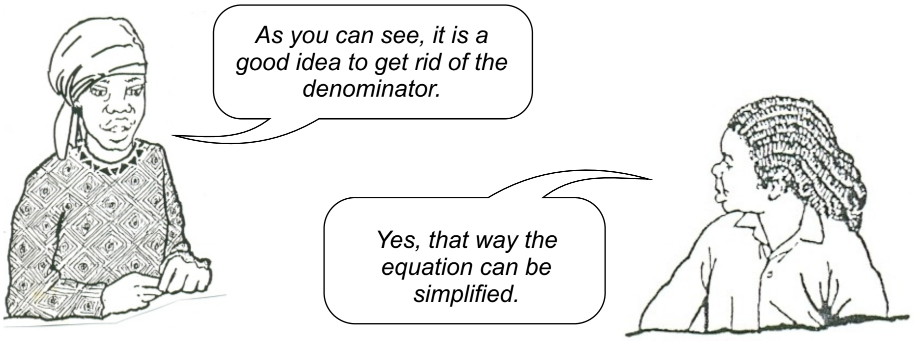
Example 5

Solve for x: $x + \frac{1}{x} = 5$

Solution

Multiply both sides of the equation by x : $x^2 + 1 = 5x$

$$\therefore x^2 - 5x + 1 = 0$$



From here, we can recognise the quadratic equation and solve it without any trouble

Do you remember the Quadratic Equation Formula?

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x^2 - 5x + 1 = 0$$

$$\therefore x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(1)}}{2(1)}$$

$$\therefore x = \frac{5 \pm \sqrt{21}}{2}$$

Example 6

Here is another one.

Solve for x: $\frac{1}{x+1} + \frac{1}{x-2} = \frac{3}{5}$

Solution

$$\frac{1}{x+1} + \frac{1}{x-2} - \frac{3}{5} = 0$$

It is a good idea here to use the lowest common denominator.

$$\frac{5(x-2) + 5(x+1) - 3(x-2)(x+1)}{5(x-2)(x+1)} = 0$$

At this point you can get rid of the denominator easily by multiplying both sides of the equation by the lowest common denominator. It is easy to do that when you have an equation.

You will then have

$$5(x-2)+5(x+1)-3(x-2)(x+1)=0$$

$$5x-10+5x+5-3(x^2-x-2)=0$$

$$10x-5-3x^2+3x+6=0$$

$$-3x^2+13x+1=0$$

$$3x^2-13x-1=0$$

Now that it's in standard form, it is simple to solve.

$3x^2-13x-1=0$

$$\therefore x=\frac{-(-13)\pm\sqrt{(-13)^2-4(3)(-1)}}{2(3)}$$

$$\therefore x=\frac{13\pm\sqrt{181}}{6}$$

See if you can do the following activity.

ACTIVITY 3

Solve the following equations:

a) $\dfrac{x\sqrt{3}}{2\sqrt{3}-x}=\dfrac{2x}{5-x\sqrt{x}}$

b) $\dfrac{y+2}{y+1}=\dfrac{y-2}{1-y}-\dfrac{4}{y-1}$

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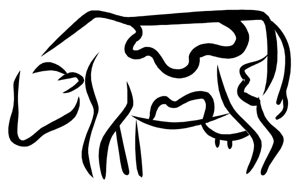
More practical applications

In Unit 1 you did some practical application problems. You may have found that they are not easy. You need more practice to give you confidence to master problems like these. Here is a practical problem for you to solve.

ACTIVITY 4

The total milk production at farm A for a one-year period is 820 tons. At farm B, with identical working conditions, the annual milk production is 1 050 tons. This is despite the fact that farm B has 60 fewer cows than farm A. The annual yield of milk per cow at farm B exceeds by 1 ton the annual yield per cow of farm A. Determine the number of cows at each farm and calculate the annual milk production per cow at each farm.



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Summary

In this lesson we found out that it is sometimes impossible to find roots of a quadratic equation through factorisation or the other method, using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Both methods usually give us roots of a quadratic equation. We realised that sometimes there are no real roots. This happens when we use the quadratic equation formula and find that we are left with a negative number under the square root sign which means that there are no real solutions to the equation.

We discovered that quadratic equations are not always obvious. Sometimes we need to do some simplifying before we can see the equation in its standard form.

Lastly, we found a way of finding the original equation when we are given roots of an equation. We found that by working backwards, we can use the roots to step through the reverse actions of solving a quadratic equation to find the equation itself from the given roots.

So we can work in both directions:
1. From the equation to its roots.
2. From the roots to the original equation.

CHECKLIST

Are you able to:

- solve various types of quadratic equations
- calculate the quadratic equation from given roots

Now, work through these exercises that will help you to check your understanding.

SELF-CHECK EXERCISE

1. Use any method to find roots of the following equations.

   a) $x^2 - 11x + 28 = 0$
   b) $4x^2 - 8x - 5 = 0$
   c) $x^2 - 14x + 45 = 0$
   d) $3x^2 + 5x + 1 = 0$
   e) $3x^2 - 3x + 1 = 0$

2. Find the original quadratic equations from the given roots.

a) $\frac{3}{2}$ and $\frac{1}{2}$

b) $\frac{3}{2}$ and $-\frac{1}{2}$

c) $-\frac{3}{2}$ and $\frac{1}{2}$

d) $-\frac{3}{2}$ and $-\frac{1}{2}$

e) 3 and 3

3. Solve for x in each of the following:

a) $2x + \frac{1}{x} = 3$

b) $\frac{x}{x-1} + \frac{2}{x-2} = 3$

4. At a certain farm, thanks to advanced methods of planting and cultivation, 680 tons of potatoes are harvested from plot A. Plot B, where advanced methods are not used, yields the same amount (680 tons). Plot B is 45 ha larger than plot A. It is found that production of potatoes using ordinary methods is 9 tons less per ha than when advanced methods are used. Determine the production of potatoes for 1 hectare for each plot of land.

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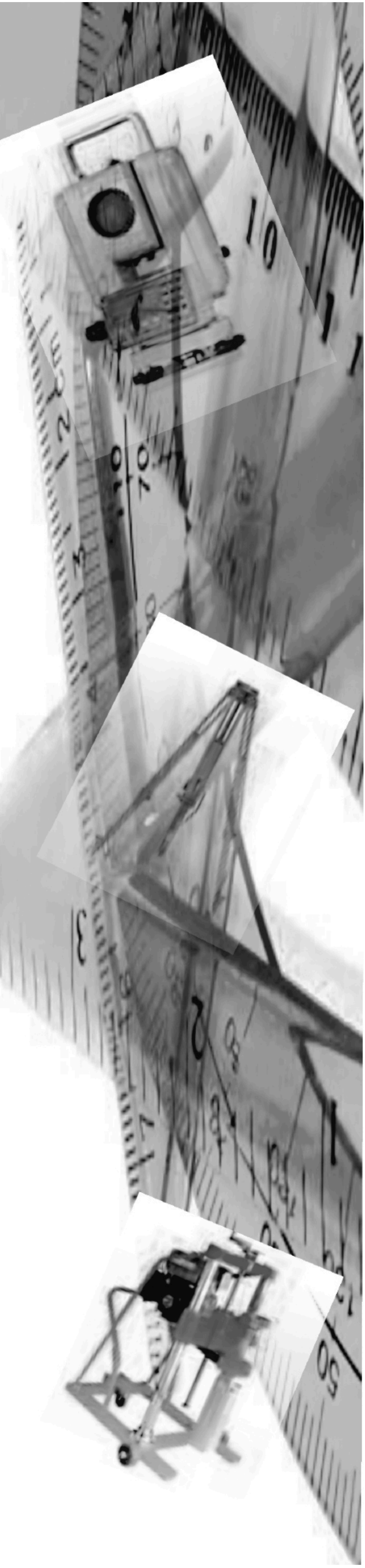
LESSON 2

Trigonometric Identities



About this lesson

Earlier you met equations that are true for all values of the variable. The equation $x + x + 5 = 2x + 1 + 4$ is an example of an equation that is true for all values of x. When you try to solve this equation you get $x = x$ or $0 = 0$, or something similar.

Such equations are called **identities**.

In trigonometry there are many identities that are very useful. Trigonometric identities are used for changing the form of expressions in order to simplify or solve for an angle.

You will need to remember the signs of ratios in different quadrants. You will also need to learn the identities and remember them.

At the end of this lesson you should be able to derive basic trigonometric identities.

In this lesson you will:

- use fundamental trigonometric identities to simplify expressions
- prove and use identities of compound angles
- prove and use identities of double angles

Fundamental trigonometric identities

To solve problems in trigonometry, we make use of many identities involving trigonometric functions. These are alternative ways of expressing a trigonometric function. For example, you have already seen some of the following:

If $A + B = 90^\circ$, then the following equations are true

$\sin A = \cos B$ $\cos B$ can be written as $\cos(90^\circ - A)$

$\sin B = \cos A = \cos(90^\circ - B)$

Fundamental trigonometric identities that are commonly used by mathematicians, scientists and engineers can be traced to two groups.

1. The quotient identities: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ You have used this before.

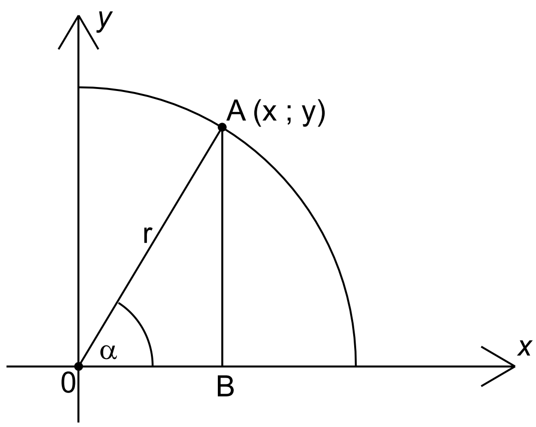
2. Squared identities (Pythagorean identities):

Pythagoras' theorem says: In a right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Do you remember the theorem of Pythagoras? We can give this theorem to justify trigonometric identities.

A right angled triangle AOB is drawn in a circle of radius *r*, with centre *O*. Vertex *A* with coordinates $(x;\ y)$ is on the circumference of this circle. Side OB is *x* units long. Side *AB* is *y* units long. $A\hat{O}B = x$

The following diagram illustrates the above description. Find all the information on the diagram before you read further.



We can now explain the theorem of Pythagoras using the given information.

You remember that $\sin\alpha = \dfrac{y}{r}$

$$\therefore y = r\sin\alpha$$
$$\cos\alpha = \frac{x}{r}$$
$$\therefore x = r\cos\alpha$$

According to the theorem of Pythagoras: $x^2 + y^2 = r^2$. When we substitute for x and y using the two equations above, we get
$(r\cos\alpha)^2 + (r\sin\alpha)^2 = r^2$

So $r^2\cos^2\alpha + r^2\sin^2\alpha = r^2$
$\therefore r^2(\sin^2\alpha + \cos^2\alpha) = r^2$

Do you remember?
Greek letters are often used for angles:

alpha = α

beta = β

theta = θ

Dividing both sides by r^2 we get $\sin^2\alpha + \cos^2\alpha = 1$

It can also be shown by starting with the left hand side of the identity:

$$LHS = \sin^2\alpha + \cos^2\alpha$$
$$= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$$
$$= \frac{y^2 + x^2}{r^2}$$
$$= \frac{r^2}{r^2} \qquad \text{or} \qquad x^2 + y^2 = r^2 \qquad \text{(Pythagoras)}$$
$$= 1$$
$$= RHS$$

This is one of the most important identities in trigonometry. From this identity we can get a number of other identities. Let us look at the following variations of this identity:

For any angle A:

$\sin^2 A + \cos^2 A = 1$
$\therefore \sin^2 A = 1 - \cos^2 A$

Similarly $\cos^2 A = 1 - \sin^2 A$

Example 1

Show that $\cos x . \tan x + \dfrac{\sin x}{\tan x} = \cos x + \sin x$

Solution

Here is how you should do such questions: Since we are given $\sin$ and $\cos$ on the right-hand side of the equation, try to change everything on the left-hand side to an expression that has $\sin$ and $\cos$ only.

***RHS** means 'right-hand side'*
***LHS** means 'left-hand side'*

$$LHS = \cos x \tan x + \frac{\sin x}{\tan x}$$

We know that $\tan x = \dfrac{\sin x}{\cos x}$, so start by making this substitution.

$$LHS = \frac{\cos x}{1} . \frac{\sin x}{\cos x} + \frac{\sin x}{\frac{\sin x}{\cos x}}$$

$$= \sin x + \frac{\sin x}{1} \times \frac{\cos x}{\sin x}$$ (to divide by a fraction, we multiply by the inverse)

$$= \sin x + \cos x$$
$$= RHS$$

Example 2

Simplify $\cos^2 y\left(1+\tan^2 y\right)+\tan^2 x.\cos^2 x+\dfrac{\sin^2 x}{\tan^2 x}$

Solution

$$\cos^2 y\left(1+\frac{\sin^2 y}{\cos^2 y}\right)+\frac{\sin^2 x}{\cos^2 x}.\cos^2 x+\frac{\sin^2 x}{\frac{\sin^2 x}{\cos^2 x}}$$ replace all tan ratios with their identity

$$= \cos^2 y + \sin^2 y + \sin^2 x + \frac{\sin^2 x}{1} \times \frac{\cos^2 x}{\sin^2 x}$$

$= 1 + \sin^2 x + \cos^2 x$ (since $\cos^2 y + \sin^2 y = 1$)
$= 1 + 1$ (since $\sin^2 x + \cos^2 x = 1$)
$= 2$

Are you following the method of answering these questions? We write the trigonometric functions in forms that we can simplify. We look for identities that will help us to simplify the answer. We substitute those identities in the equation or expression and use algebra to simplify. Try the following activity.

ACTIVITY 1

Prove the following identities:

1. $\tan A.\cos A = \sin A$

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2. $1-\left(\sin A-\cos A\right)\left(\sin A+\cos A\right)= 2\sin^2 A$

Identities of compound angles

For any two angles α and β, the following identities are true:

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

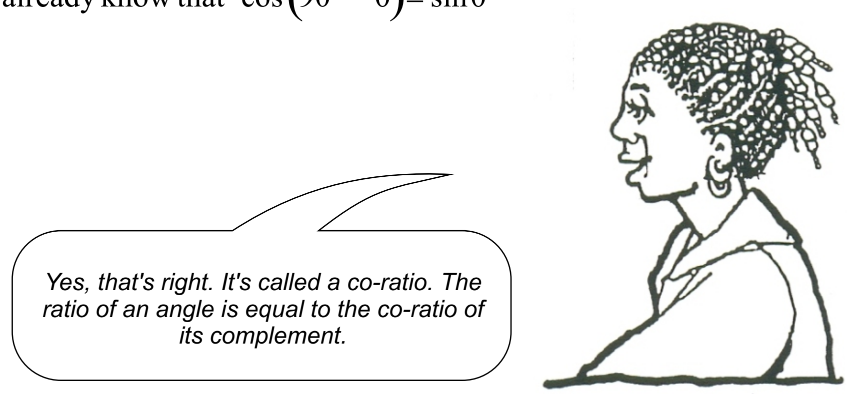
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

*A **compound angle** is an angle that is expressed as a sum of, or the difference between two angles. Any angle can be a compound angle.*

The proof of the first of these identities is complex and beyond the scope of this lesson, so let's rather use it in a situation where we have an alternative method of simplification just to prove to ourselves that the identity holds true.

We already know that $\cos(90^\circ - \theta) = \sin\theta$



Now let's use the compound angle identity to check if we get the same outcome.

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\begin{aligned} \therefore \cos(90^\circ - \theta) &= \cos 90^\circ \cos\theta + \sin 90^\circ \sin\theta \\ &= (0).\cos\theta + (1)\sin\theta \\ &= \sin\theta \end{aligned}$$

$\cos 90^\circ = 0$

$\sin 90^\circ = 1$

which we know to be true.

To derive the identity for $\cos(\alpha + \beta)$ you will need to remember a few facts from the previous unit.

When we were dealing with negative angles and quadrants we found that $\cos(-\theta) = \cos\theta$

Also, $\sin(-\theta) = -\sin\theta$

We can now proceed as follows:

$\cos(\alpha+\beta) = \cos(\alpha-(-\beta))$

$= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$ (using the identity we derived above)

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

At the start of this lesson we saw that $\sin\alpha = \cos(90^\circ - \alpha)$

We can use this to derive the formula for sin (α + β), as follows:

$\sin(\alpha+\beta) = \cos(90^\circ - (\alpha+\beta))$

$= \cos((90^\circ - \alpha) - \beta)$

$= \cos(90^\circ - \alpha)\cos\beta + \sin(90^\circ - \alpha)\sin\beta$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

And $\sin(\alpha-\beta) = \sin(\alpha+(-\beta))$ to derive the formula for sin (α – β)

$= \sin\alpha\cos(-\beta) + \sin(-\beta)\cos\alpha$ (using the identity we have proven above)

$= \sin\alpha\cos\beta - \sin\beta\cos\alpha$

Redution Formula:

$\sin 101^\circ = \sin(180^\circ - 79^\circ)$

$= \sin 79^\circ$

co-ratios:

$\cos 56^\circ = \cos(90^\circ - 34^\circ)$

$= \sin 34^\circ$

Example 3

Simplify to one trigonometric ratio: $\cos 79^\circ \cos 34^\circ + \sin 101^\circ \cos 56^\circ$

Solution

$\cos 79^\circ \cos 34^\circ + \sin 101^\circ \cos 56^\circ$

$= \cos 79^\circ \cos 34^\circ + \sin 79^\circ \sin 34^\circ$

$= \cos(79^\circ - 34^\circ)$

$= \cos 45^\circ$

Try the following activity. Remember, all that is necessary is to use the identities.

ACTIVITY 2

1. Prove the following identities, using compound angle identities:

   a) $\sin(90^\circ - \theta) = \cos\theta$

   b) $\cos(90^\circ + \theta) = -\sin\theta$

2. Simplify:

   a) $\cos A \sin B + \sin A \cos B$

   b) $\cos 50^\circ \cos 40^\circ - \sin 50^\circ \sin 40^\circ$

   c) $-\cos 3x \cos x - \sin 3x \sin x$

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Double Angle Identities

Double angles are also compound angles. The only difference is that, instead of having $A+B$, we have $A=B$. Then, the sum of angles $A+B$ becomes $A+A=2A$

$\sin 2A \neq 2\sin A$
$\cos 2A \neq 2\cos A$

ACTIVITY 3

Expand and simplify the following using the compound angle identities we have just done:

1. $\cos 2\alpha$

2. $\sin 2\alpha$

Hint: $\sin 2\alpha = \sin(\alpha+\alpha)$

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From this activity, we can see the double angle identities.

$\sin 2\theta = 2\sin\theta\, cos\,\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

Let's look at how we use these identities.

Example 4

Prove the following identity: $\dfrac{1-\cos 2\theta - \sin\theta}{\sin 2\theta - \cos\theta} = \tan\theta$

Solution

This is difficult to simplify since the angles are not all the same size. To get them to all be of equal size, we use the double angle identities.

$$LHS = \frac{1-(\cos^2\theta - \sin^2\theta) - \sin\theta}{2\sin\theta\cos\theta - \cos\theta}$$

$$= \frac{1-\cos^2\theta + \sin^2\theta - \sin\theta}{\cos\theta(2\sin\theta - 1)}$$

We find now that the numerator is difficult to simplify, so we use the Pythagorean Identity.

$$= \frac{1-(1-\sin^2\theta) + \sin^2\theta - \sin\theta}{\cos\theta(2\sin\theta - 1)}$$

$$= \frac{1-1+\sin^2\theta + \sin^2\theta - \sin\theta}{\cos\theta(2\sin\theta - 1)}$$

$$= \frac{2\sin^2\theta - \sin\theta}{\cos\theta(2\sin\theta - 1)}$$

$$= \frac{\sin\theta(2\sin\theta - 1)}{\cos\theta(2\sin\theta - 1)}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta$$

$$= RHS$$

Although these identities can get quite complex, we just need to take one step at a time, using the different identities as and when we need them. Let's try another example.

Example 5
Prove the identity:

$$\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}=\frac{2}{\sin 2A}$$

Solution

$$\begin{aligned}
LHS &= \frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\\
&= \frac{\sin^2 A+\cos^2 A}{\sin A\cos A}\\
&= \frac{1}{\sin A\cos A}\\
&= \frac{1}{\frac{1}{2}\sin 2A}\\
&= \frac{2}{\sin 2A}\\
&= RHS
\end{aligned}$$

$$2\sin A\cos A=\sin 2A$$
$$\therefore \sin A\cos A=\tfrac{1}{2}\sin 2A$$

It's time to try a few on your own.

ACTIVITY 4

Prove the following identities.

1. $\dfrac{\sin 2A}{1+\cos 2A}=\tan A$

2. $\dfrac{\sin 3x+\sin x}{1+\cos 2x}=2\sin x$

   This identity is tricky. You will need to use all your skills. It requires the use of a Compound angle identity, two Double angle identities and the Pythagorean identity.

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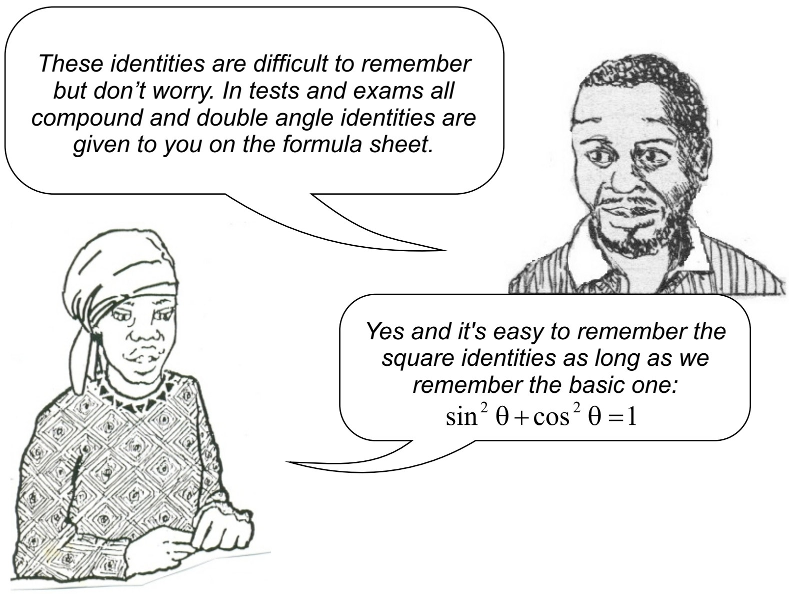
How did that activity go?

Summary

In this lesson you learnt to deal with trigonometric identities. Here is a table containing all the identities:

QUOTIENT IDENTITY	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
SQUARE IDENTITIES	$\sin^2\theta + \cos^2\theta = 1$
	$\sin^2\theta = 1 - \cos^2\theta$
	$\cos^2\theta = 1 - \sin^2\theta$
COMPOUND ANGLE IDENTITIES	$\cos(A - B) = \cos A\cos B + \sin A\sin B$
	$\cos(A + B) = \cos A\cos B - \sin A\sin B$
	$\sin(A - B) = \sin A\cos B - \sin B\cos A$
	$\sin(A + B) = \sin A\cos B + \sin B\cos A$
DOUBLE ANGLE IDENTITIES	$\sin 2A = 2\sin A\cos A$
	$\cos 2A = \cos^2 A - \sin^2 B$



CHECKLIST

Are you able to:

- recognise where to use trigonometric identities
- write down trigonometric identities that can be used to simplify expressions
- use trigonometric identities to prove statements
- use trigonometric identities to simplify expressions

If you have a problem, discuss it with friends or your tutor. Discussions are important in the learning process, even if you think you understand. Now try the following self-check exercises.

SELF-CHECK EXERCISE

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1. Prove that $(\cos x + \sin x)^2 + (\cos x - \sin x)^2 = 2$

2. Prove that $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

3. Prove that $\dfrac{\sin A \sin 2A}{\cos A} + \cos 2A = 1$

LESSON 3

Trigonometric Equations

About this lesson

In the previous lesson you learnt about trigonometric identities. Lessons 3 and 4 use trigonometric identities in the solution of trigonometric equations. The unknown in a trigonometric equation is the angle. You will need a calculator for these tasks.

For this lesson you need to know the identities you have come across so far.

In this lesson you will:

- revise identities
- solve simple trigonometric equations
- find the general solution for equations which indicate all the possible angles
- use identities to solve trigonometric equations
- solve the linear equation in $\cos x$ and $\sin x$.

Revision of identities

Before we go into the solution of equations, you may need to revise some identities. First here is an example which will give you an idea of what to do.

Example 1
Prove the identity $1 + \cos 2\alpha + 2\sin^2\alpha = 2$

Solution
We know that $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$
\begin{aligned}
LHS &= 1 + \cos 2\alpha + 2\sin^2\alpha \\
&= 1 + \cos^2\alpha - \sin^2\alpha + 2\sin^2\alpha \\
&= 1 + \cos^2\alpha + \sin^2\alpha \\
&= 1 + 1 \\
&= 2
\end{aligned}
$$

Now do the following examples:

ACTIVITY 1

Prove the equality $\dfrac{\sin 3\alpha}{\sin\alpha} - \dfrac{\cos 3\alpha}{\cos\alpha} = 2$

Hint: write the LHS with a common denominator and then use the identity $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ from right to left.

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You will achieve the end result when you are able to get the left hand side equal to the right hand side.

Solving trigonometric equations

A trigonometric equation is an equation that has trigonometric ratios in it.

Example 2
Solve for α in $\sin\alpha = 0{,}5$

Solution
Solving an equation like this means finding all values of alpha for which the equation will be true. How would you answer this simple question:

For which angle is the sine equal to $0{,}5$?
Easy, isn't it? $\sin 30^\circ = 0{,}5$. We know this already from Unit 2. So $\alpha = 30^\circ$

We can also find it using a calculator:
$\sin^{-1}(0{,}5) = 30^\circ$ (shift $\rightarrow$ sin $\rightarrow$ 0.5 $\rightarrow$ =)

After Unit 2 you are familiar with using a calculator for trigonometric functions. For practice, try the following activity.

ACTIVITY 2

Solve the following simple equations:

a) $\sin x = 0{,}57$
b) $\cos x = 0{,}866$
c) $\sin x = -0{,}866$
d) $\tan\alpha = 1$
e) $\cos y = -1$
f) $\tan\beta = 1{,}73$
g) $\sin x = 0{,}866$
h) $\cos A = 0{,}707$
i) $\tan B = -0{,}577$
j) $\sin x = 0{,}707$

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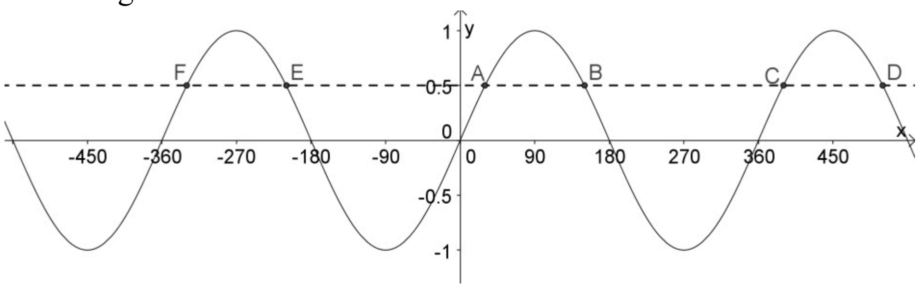
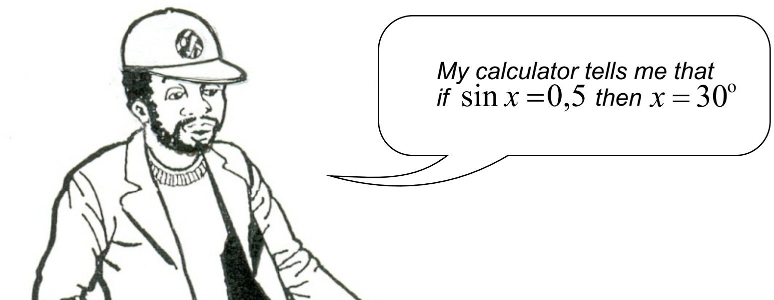
Finding all angles

The answers given in the feedback to Activity 2 are not complete. In fact you may not know how to get some of the answers. Do you know why? Let us explain with the use of a graph of the sine function. Look at the diagram below.



This diagram shows the graph of a sine function with values of angles (the domain) from -540° to 540°. If we draw a line $y = 0{,}5$ on the same axes we find that there are a number of points where the two graphs intersect. Let us discuss these intersections.

We are looking at the intersections between two graphs, $y = \sin x$ and $y = 0{,}5$. Therefore, the points of intersection will be points which satisfy the equation $\sin x = 0{,}5$. Between the angles 0° and 360° there are **4** such points at A, B, C and D. In the domain of -540° and 540°. There are **6** points of intersection. What are these points?



Yes, by solving the simple equation, we find that $x = 30^{\circ}$ and we can see this at point A when looking at the graph. Due to the symmetrical nature of the curve, we can see that point B is 30° to the left of 180° which is 150°. This will always be the case with equations involving sine. We determine the first two answers by using the solution that is given to us from the calculator as well as 180° less that solution.

$$x = \sin^{-1}(0{,}5) \quad \text{or} \quad x = 180^{\circ} - \sin^{-1}(0{,}5)$$

$$\therefore x = 30^{\circ} \qquad \therefore x = 150^{\circ}$$

This is still not enough. We can see that there are several other intersections on the graph and the greater we make the domain, the more answers there will be.

An understanding of repetition and revolutions helps us to remember that the values of trigonometric functions repeat themselves. If we add 360° to 30° we get 390°, at point C. If we add 360° to 150° we get 510°, at point D. The intersections at points E and F are obtained by subtracting 360° from both 30° and 150° respectively.

The equation of $\sin x = 0{,}5$ is satisfied by the 30° and 150°, 360° added to both 30° and 150°, 360° subtracted from both 30° and 150° and any multiple of 360° both added or subtracted to and from 30° and 150°, giving us multiple answers.

In this form of writing, Z stands for all positive and negative whole numbers. These numbers are called integers.

That is why there are many angles where the sine value is equal to 0,5. We write the answer in a general form as follows:

If $\sin x = 0{,}5$, then $x = 30^\circ + k.360^\circ$ or $x = \left(180^\circ - 30^\circ\right) + k.360$ where k is any integer.

Many mathematics books will write the answer like this:
$x = 30^\circ + k.360^\circ$ or $x = 150^\circ + k.360$ where $k \in Z$

Example 3

Solve for x in $\sin x = -0{,}267$ rounded off to 2 decimal places.

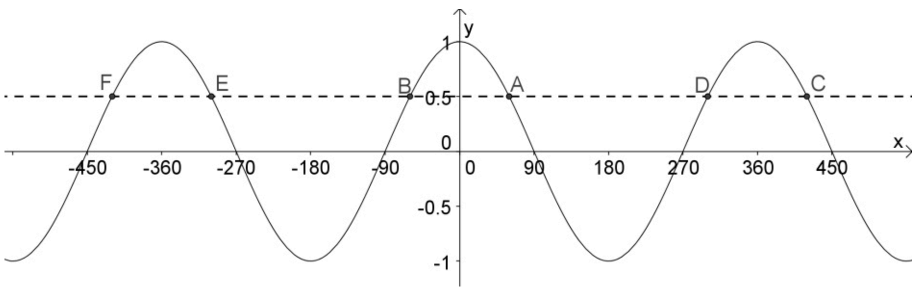
calculator strokes:
shift→sin→ −0.267→=

Solution

$x = -15{,}49^\circ + k.360$
or $x = 180^\circ - \left(-15{,}49^\circ\right) + k.360 = 195{,}49^\circ + k.360^\circ$ where $k \in Z$

This is referred to as the general solution.

What about equations involving cos? Let's have a look at the graph of the cosine function.



Here we have the functions $y = \cos x$ and $y = 0.5$. Again we can see that there are several points of intersection, indicating that there are several solutions to the equation $\cos x = 0.5$. Using our calculation we find that $x = 60^\circ$ is one solution to the equation, at point A.

Again, the symmetrical nature of the curve shows us that a second solution exists at point B, which is the same distance from the y −axis at the point A. This is $x = -60^\circ$ and as happened with sine, this too will always be the case with cosine.

We can always determine the first two solutions by using the calculator solution and the negative of that solution. As with sine, we then add and subtract multiples of 360^{o} to get the remaining solutions.

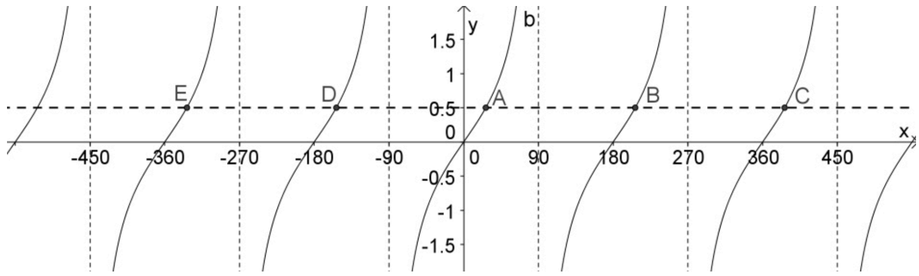
So, if $\cos x = 0{,}5$, then $x = 60^{o} + k360^{o}$ or $x = -60^{o} + k360^{o}$ where $k \in Z$

Example 4
Solve for A in $\cos A = -0{,}12$ rounded off to 2 decimal places.

calculator strokes:
shift→*sin*→ -0.12 →=

Solution
$x = 90{,}69^{o} + k.360$ or $x = -90{,}69 + k.360$ where $k \in Z$

All that is left is to look at the tangent function.



The tangent curve differs from the other two in that it repeats every 180^{o} instead of every 360^{o}. This makes solving equations involving tan much easier since there is only one initial solution, and then multiples of 180^{o} are added and subtracted.

So if, $\tan x = 0{,}5$ then $x = 26{,}57^{o} + k.180^{o}$ when $k \in Z$

Example 5
Solve for *y*, rounded off to two decimal places in $3\tan x - 2 = 0$

Solution
First we need to isolate the trig ratio with its angle $3\tan x = 2$

$$\therefore \tan x = \frac{2}{3}$$

Then we find the general solution $\therefore x = 33{,}69^{o} + k.180$ where $k \in Z$

In summary, trig equations are dealt with as follows:

$\sin\theta = p$ then $\theta = \sin^{-1}(p) + k.360$ or $\theta = 180^{o} - \sin^{-1}(p) + k.360^{o}$ where $k \in Z$

$\cos\theta = m$ then $\theta = \cos^{-1}(m) + k.360$ or $\theta = -\cos^{-1}(m) + k.360^{o}$ where $k \in Z$

$\tan\theta = q$ then $\theta = \tan^{-1}(q) + k.180$ where $k \in Z$

Do you understand what is written here in this section so far? If you do not, read it again.

Now try the following activity.

ACTIVITY 3

Find the general solution for each of the following equations:

1. $-2\sin x + 3 = 0$

2. $3\cos(x - 20) + 1 = 0$

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3. $\tan 2x = 3$

Trigonometric Equations

Now you have everything you need to solve trigonometric equations. Here is an example of the type of question you may come across.

Example 6
Solve the equation $\sin^2 x - \cos^2 x = 0{,}5$

Solution
We would like to express the equation in only one unknown, or at least a product of 2 or more factors. We therefore try to simplify what we have using identities.

$\sin^2 x - \cos^2 x = 0{,}5$

$\therefore -\left(\cos^2 x - \sin^2 x\right) = 0{,}5$ (using a switch-round)

$\therefore -\cos 2x = 0{,}5$

$\therefore \cos 2x = -0{,}5$ (using the double angle identity for $\cos 2A$)

$\therefore 2x = 120° + k.360°$ or $2x = -120° + k360°$ (solve for the angle first)

$\therefore x = 60° + k180°$ or $\therefore x = -60° + k180°$ (then solve for x)

ACTIVITY 4

Solve the following equations:

1. $\sin x \cos x = \dfrac{\sqrt{3}}{4}$

2. $\sin x = \sin 2x$

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3. $\cos 2x = \cos x$

Linear equations in $\cos x$ *and* $\sin x$

Example 7
Solve for x in $\sin x = \cos x$

Solution
In this example, we have a single ratio on each side with no squares or double angles. Since the angles of both ratios in this example are the same size (both equal to x), the easiest way to proceed is to use the quotient identity, which is $\dfrac{\sin A}{\cos A} = \tan A$

If we divide both sides of this equation by $\cos x$ we will see how it works.

$$\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$$

$\therefore \tan x = 1$ This then becomes a really simple equation.
$\therefore x = 45^\circ + k.180^\circ$ where $k \in Z$

Example 8
Solve for α in $\cos 2\alpha = \sin 2\alpha$

Solution
Even though this example presents two double angles that could be broken up using double angle identities, the angles of the two ratios are the same (both equal to 2α), and we have both a sine ratio and a cosine ratio. This is best dealt with in the same manner as the previous example.

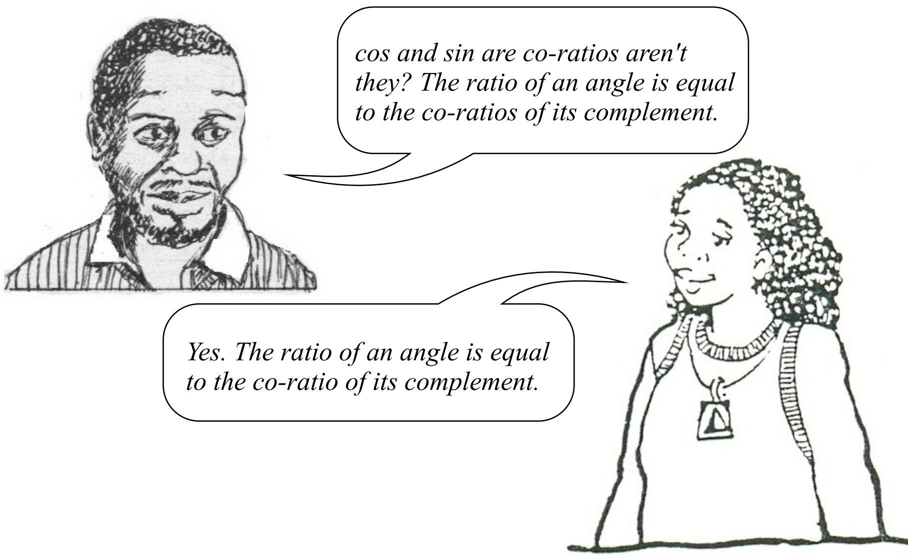
$$\frac{\cos 2\alpha}{\cos 2\alpha} = \frac{\sin 2\alpha}{\cos 2\alpha}$$

$\therefore 1 = \tan 2\alpha$
$\therefore 2\alpha = 45 + k.360^\circ$
$\therefore \alpha = 22{,}5^\circ + k.90$ where $k \in Z$

This method is both quick and simple but only helps us if the angles of both sine and cosine are the same. We need another method to deal with the next example.

Example 9
Solve for x in the following equation: $\sin 2x = \cos(x + 10^\circ)$

Solution
A common error made when solving equations like these, is to rush in and use a double angle identity on the left hand side and the compound angle identity on the right hand side. This gives us three separate terms that we cannot simplify or factorise. This type of equation requires us to remember co-ratios.



By using co-ratios, we can change one side of this equation to have the same ratio as the other. Since $\sin 2x = \cos(90^\circ - 2x)$, we have an equation in cos only: $\cos(90^\circ - 2x) = \cos(x + 10^\circ)$

It makes logical sense that if the cos of one angle is equal to the cos of another angle, then the two angles could be equal to each other. Be careful though, don't forget about the second angle that needs to be taken into account.

$\therefore 90^\circ - 2x = x + 10^\circ + k.360^\circ$ or $90^\circ - 2x = -(x + 10^\circ) + k.360^\circ$

$\therefore -3x = -80^\circ + k.360^\circ$ or $-x = -100^\circ + k.360^\circ$

$\therefore x = 26{,}67^\circ + k.120^\circ$ or $x = 100^\circ + k.360^\circ$ where $k \in Z$

Try the activity below.

ACTIVITY 5

1. $\cos(2x + 15^\circ) = \sin x$

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2. $\sin 2x \cos 20 + \cos 2x \sin 200 = \cos 2x$

Summary

In this lesson you learnt that a trigonometric equation may have more than one answer. You were also shown that although the calculator will give you one angle, you have to find the other angles yourself using the general solution. Finally you saw how complex equations can be and what you need to do is remember the different approaches.

CHECKLIST

Are you able to:

- solve simple trigonometric equations using a calculator
- find the general solution to more complex equations
- solve trigonometric equations using identities.

Now do the following exercise.

SELF-CHECK EXERCISE

Find the general solution for each of the following equations:

1. $2\sin^2 x = \cos 2x$

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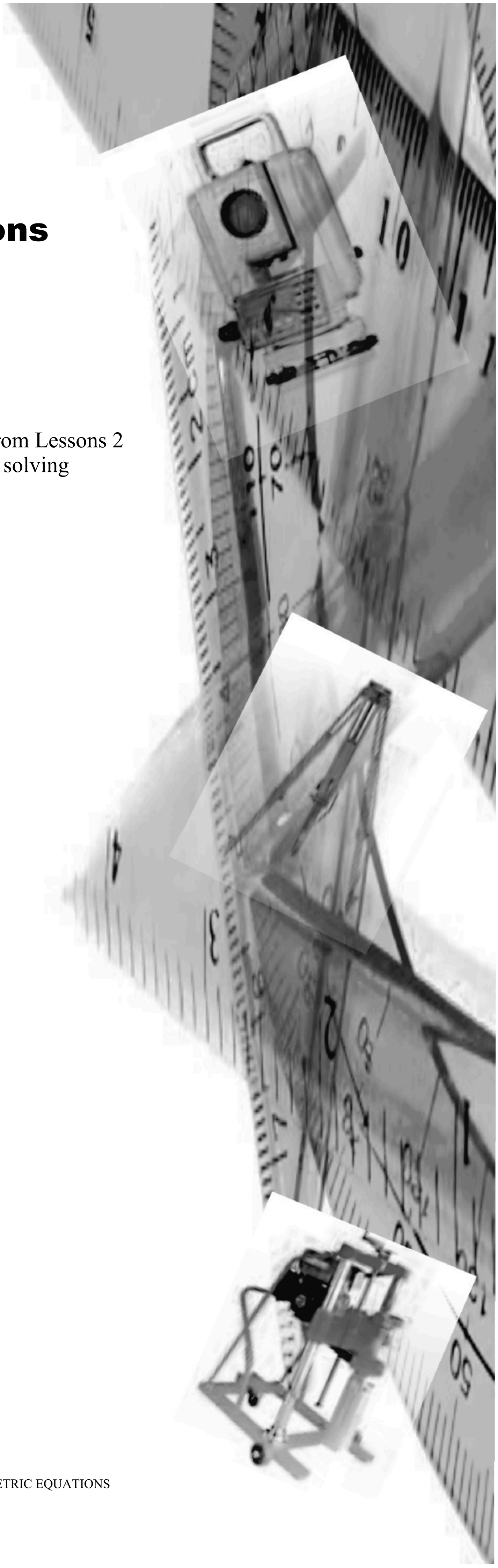
2. $5\sin x - 2 = 1 - \cos 2x$

LESSON 4

More Trigonometric Equations



About this lesson

In this lesson you will need to remember the identities from Lessons 2 and 3. You will also have to remember different ways of solving equations.

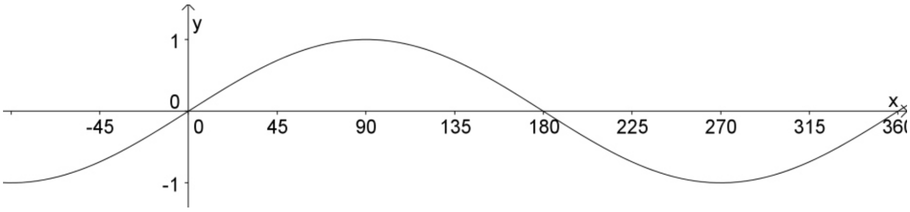
In this lesson you will:

- express answers in given intervals
- solve trigonometric equations without a calculator
- use identities involving square roots

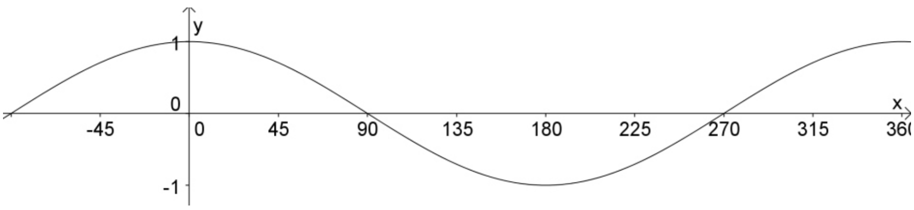
Solving equations without a calculator

So far we have used our calculators to find values of trigonometric ratios. There are ratios that are so commonly used that you do not need to use a calculator for them. These are ratios that involve $30^{o}, 45^{o}, 60^{o}$ and 90^{o} and their multiples. These angles are called 'special angles'.

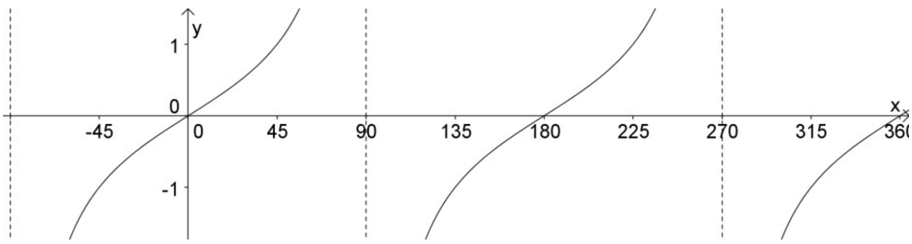
We already know that $\sin 0^{o} = 0$ while $\cos 0^{o} = 1$. We know also that $\sin 90^{o} = 1$ while $\cos 90^{o} = 0$. This can be easily seen on the graphs of the functions $y = \sin x$ and $y = \cos x$.



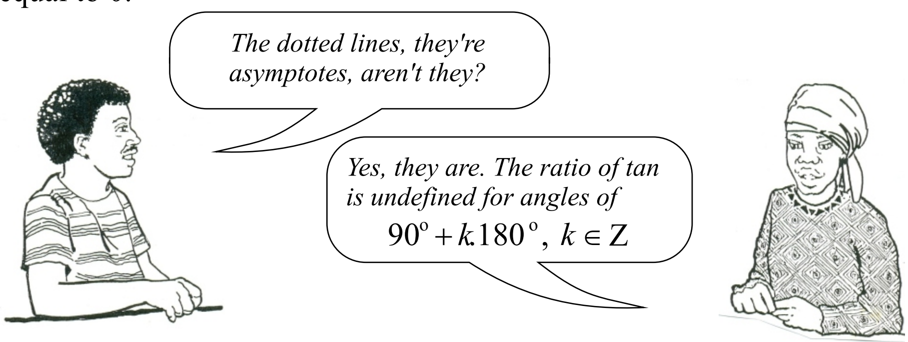
From $y = \sin x$ above we see that $\sin 0^{o}$, $\sin 180^{o}$ and $\sin 360^{o}$ are all equal to 0, while $\sin 90^{o} = 1$ and $\sin 270^{o} = -1$



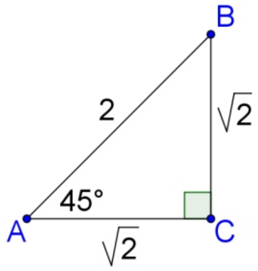
From $y = \cos x$ above we see that $\cos 0^{o}$ and $\cos 360^{o}$ are both equal to **1**, while $\sin 90^{o}$ and $\cos 270^{o}$ are equal to **0**.



From $y = \tan x$ above, we see that $\tan 0^{o}$, $\tan 180^{o}$ and $\tan 360^{o}$ are all equal to 0.



Certain ratios of angles between 0° and 90° can easily be found. Look at the following triangle.



In triangle ABC, angles A and B are both 45°. Since the two angles are equal, the sides opposite them will be equal, making it an isosceles triangle. If we make the hypotenuse 2 units in length, then the other two sides would be $\sqrt{2}$. We determine this using the Theorem of Pythagoras.

$$x^2 + x^2 = 2^2$$
$$2x^2 = 4$$
$$x^2 = 2$$
$$x = \sqrt{2}$$

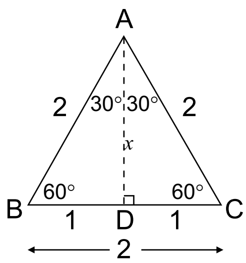
The proportion, therefore, of any right angled isosceles triangle would be $\sqrt{2} : \sqrt{2} : 2$. Using these proportions we can see that:

$$\sin 45^{\circ} = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^{\circ} = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^{\circ} = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

In any equilateral triangle all angles are 60° and all sides are equal.
Suppose AB = BC = AC = 2.
Then AD ⊥ BC bisects BC and BD = DC = 1
Also AD bisects Â so BÂD = CÂD = 30°



In the right-angled triangle ABD:

$$1^2 + x^2 = 2^2$$
$$x^2 = 3$$
$$x = \sqrt{3}$$

The proportion, therefore, of any right-angled triangle with angles of 60° and 30° would be $\sqrt{3}:1:2$.

Using these proportions we can see that in Δ ABD:

$$\sin 30^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \qquad \sin 60^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}$$

$$\tan 30^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}} \qquad \tan 60^\circ = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Example 1

Solve for α without the use of a calculator: $2\cos\alpha - 1 = 0$

$2\cos\alpha - 1 = 0$

$\therefore \cos\alpha = \frac{1}{2}$

At this stage we need to find which angle would give us the ratio of $\frac{1}{2}$ when dividing the adjacent side by the hypotenuse. If we look back at our triangles we see that the angle is 60°. As with all equations involving cos, we need to consider the other angle.

$\therefore \alpha = 60^\circ + k.360^\circ$ or $\alpha = -60^\circ + k.360^\circ$ where $k \in Z$

Any equation involving the ratios $(\frac{1}{2}; \frac{2}{1}; \frac{\sqrt{3}}{2}; \frac{2}{\sqrt{3}}; \frac{\sqrt{2}}{2}; \frac{2}{\sqrt{2}})$

can be solved in this way. The sin, cos and tan of angles 30°, 45° and 60° can be evaluated without using a calculator.

ACTIVITY 1

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Solve for the unknown without using a calculator: $2\sin\theta - \sqrt{3} = 0$

Expressing answers in given intervals

Sometimes you are told that your answer has to be within certain given limits instead of just asking for the general solution. Let us consider an example.

Example 2

Solve for the unknown angle: $2\sin A - 1 = 0$ where $A \in [-360^\circ, 180^\circ]$

Solution

$2\sin A - 1 = 0$

$\therefore \sin A = \frac{1}{2}$

$\therefore A = 30^\circ + k.360$ or $A = 180^\circ - 30^\circ + k.360^\circ$

$= 150^\circ + k.360^\circ$ where $k \in Z$

This stage of the solution is referred to as the General Solution. In this example we need to find all the answers within the given limit. We use the general solution to do this. We start with the first answer and we ask ourselves whether this answer fits into the given limit. It does.

Now we need to add and subtract multiples of 360° to our answer of 30° until our answers fall beyond the limit.

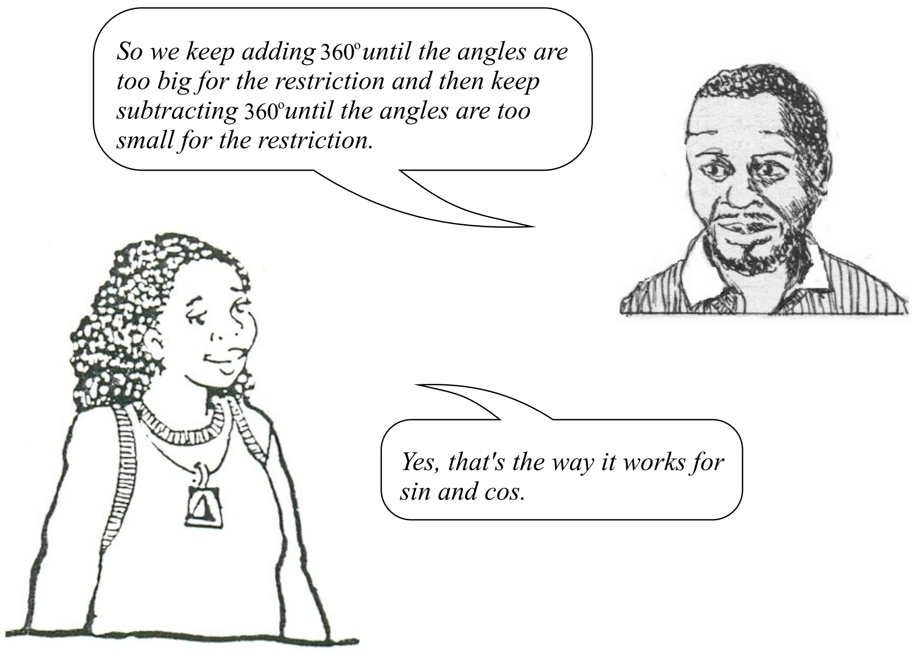
$30^\circ + 360^\circ = 390^\circ$ is too big, so we needn't continue adding to it.
$30^\circ - 360^\circ = -330^\circ$ is within the limit, so we subtract another 360°
$-330 - 360^\circ = -690^\circ$ is too small so we needn't continue.

Our next answer was 150°. Does this answer fit into our limit? Yes it does.

Now we need to add and subtract multiples of 360° from our answer of 150° until our answers fall beyond the limit.

$150^\circ + 360^\circ = 510^\circ$ is too big, so we needn't continue adding to it.
$150^\circ - 360^\circ = -210^\circ$ is within the limit, so we subtract another 360°
$-210^\circ - 360^\circ = -570^\circ$ is too small so we needn't continue.

Now we need to list all the answers that were within the limit.
$\therefore x = 30^\circ,\ 150^\circ,\ -210^\circ,\ -330^\circ$



Things are a little different for tan since the tan function repeats every 180° unlike the sine and cosine functions which repeat every 360°.

Let's look at another example.

Example 3
Solve the following equation: $3\tan\theta = 9{,}6$ for $180^\circ < \theta < 360^\circ$

Solution

$\tan\theta = \dfrac{9{,}6}{3}$

$\therefore \theta = 72{,}65^\circ + k.180^\circ$ where $k \in Z$

This is the general solution. We still need to find the angles that fall within the limit.

Does $72{,}65^\circ$ fit? - No, it doesn't.

Add multiples of 180°.

Does $72{,}65^\circ + 180^\circ = 252{,}65^\circ$ fit? Yes, it does.

Does $252{,}65^\circ + 180^\circ = 432{,}65^\circ$ fit? No, therefore we needn't continue.

Subtract multiples of 180°

Does $72{,}65^\circ - 180^\circ = -107{,}35^\circ$ fit? No, therefore we needn't continue.
$\therefore \theta = 252{,}65^\circ$ is the only solution.

Now try the following activity.

ACTIVITY 2

Solve for the unknown in the following:

1. $3\tan\theta + 1 = 0$ for $\theta \in [-360^\circ ; 180^\circ]$
2. $\dfrac{\sin 2\theta}{\cos\theta} - 1 = 0$ for $\theta \in [-180^\circ ; 360^\circ]$
3. $\sin 2\theta + 2\sin\theta\, 2\cos\theta = 20$ for $\theta \in [-360^\circ; 0^\circ]$

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More complicated equations

The first examples here require factorisation. Do you remember factorisation? If you do not, please revise previous units.

Example 4

Give the general solution for the following: $\cos^4 x - \cos^2 x = 0$

Solution

If you look carefully at this equation you will see that it can be solved using algebra and trigonometric identities together.

$\cos^4 x - \cos^2 x = 0$

$\cos^2 x\left(\cos^2 x - 1\right) = 0$ (taking out a common factor)

$\therefore \cos^2 x = 0$ or $\cos^2 x = 1$

$\therefore \cos x = 0$ or $\cos x = \pm 1$

$\therefore \cos = \pm 90^\circ + k.360^\circ$ or $x = \pm 0^\circ + k.360$ or $x = \pm 180^\circ + k.360^\circ$ where $k \in Z$

Both **1** and **–1** need to be considered as we have square rooted both sides of the equation in order to solve it. This action must always result in both the positive and negative ratios being followed.

Example 5

Solve for *x*: $\dfrac{1+\cos x}{\cos \frac{x}{2}} = \dfrac{\sin x}{-1 \cos x}$

Solution

First step is to try to get around working with all the fractions. The best way to do this is to find the lowest common denominator, in this case,

$\cos\frac{x}{2}(1 - \cos x)$

$\therefore (1 + \cos x)(1 - \cos x) = \sin x \cos\frac{x}{2}$ (multiplying through by a common denominator)

$\therefore 1 - \cos^2 x = \sin x \cos^2 x$ (getting rid of brackets)

$\therefore \sin^2 x = \sin x \cos\frac{x}{2}$ (using the Pythagorean Identity)

$\therefore \sin^2 x - \sin x \cos\frac{x}{2} = 0$

$\therefore \sin x\left(\sin x - \cos\frac{x}{2}\right) = 0$ (taking out a common factor)

$\therefore \sin x = 0$ or $\sin x = \cos\frac{x}{2}$

Did you notice how much algebraic knowledge you were using in order to solve this trigonometric equation?

Now we deal with each equation separately. The one equation is a simple equation in terms of sine, the other is in terms of both sine and cosine where the angles are not the same size. Remember you need to use co-ratios to solve both sides in terms of cos or both sides on terms of sin.

$\therefore x = 0^o + k360^o$ or $x = 180^o + k.360^o$ or $\cos(90 - x) = \cos\dfrac{x}{2}$

You could have chosen to change the ratio on the right hand side to $\sin\left(90 - \dfrac{x}{2}\right)$ and get an equation in terms of sine only.

This will give you exactly the same answers.

$$\therefore 90 - x = \frac{x}{2} + k.360^o \quad \text{or} \quad 90 - x = -\frac{x}{2} + k.360^o$$

$$\therefore 180^o - 2x = x + k.720^o \text{ or } \quad 180^o - 2x = -x + k.720^o$$
$$\therefore -3x = -180^o + k.720^o \quad \text{or} \quad -x = -180^o + k.720^o$$
$$\therefore x = 60^o + k.240^o \quad \text{or} \quad x = 180^o + k.720^o \text{ where } k \in Z$$

Since this equation involves fractions where the unknown forms part of the denominator, we need to make sure that none of our solutions will result in a zero denominator. Remember that fractions with a zero denominator are undefined. Let's check all the answers in the example above.

If $x = 0^o$ then $1 - \cos x = 1 - \cos 0$
$$= 1 - 1$$
$$= 0$$

The denominator on the right hand side would be zero and we therefore need to discard this answer.

If $x = 180^o$ then $\cos\dfrac{x}{2} = \cos\dfrac{180^o}{2}$
$$= \cos 90^o$$
$$= 0$$

The denominator on the left hand side would be zero and we therefore need to discard this answer.

The only answer remaining is therefore $x = 60^o + k240^o$

Try the following activity to check that you understand what we have just done.

ACTIVITY 3

Solve the following equations, giving the general solution for each:

1. $2\sin x \tan x - \sin x - 2\tan x + 1 = 0$

2. $\sin^2 x - 10\sin x \cos x + 21\cos^2 x = 0$

3. $\sin(x - 90^o) - \sin 90^o = \sin(x + 90^o)$

ANSWERS ON PAGE 62

Summary

In this lesson we have evaluated trigonometric ratios of the special angles without using a calculator. This gives an exact solution rather than a decimal approximation.

The solutions to trigonometric equations might be required within given intervals. Solutions can also be expressed in a general form that gives all possible solutions to the equation.

To solve trigonometric equations successfully, you need to remember algebraic techniques for solving equations. Eliminating fractions, and factorising can transform problems into two or three simple trigonometric equations that are easily solved.

CHECKLIST

Are you able to:

- find the values of trigonometric ratios of the special angles without using a calculator
- express solutions to trigonometric equations in the general form
- find solutions to trigonometric equations for required intervals
- use algebraic techniques to simplify complicated trigonometric equations.

Having made sure that you understand the above, do the following self-check exercises.

SELF-CHECK EXERCISE

1. Without using your calculator, write down the values of the following:

   a) $\sin^2 60^\circ + \tan^2 45^\circ$
   b) $1 - \sin^2 30^\circ$

2. Solve the following, without the use of a calculator, for the angles between -180° and 360°:

   a) $2\cos^2\theta - 1 = 0$
   b) $\tan\theta \sin 2\theta = 0$

3. Solve the following for the angles between -360° and 360°

a) $\dfrac{\sqrt{1+\cos\theta}}{2} = \sin\theta$

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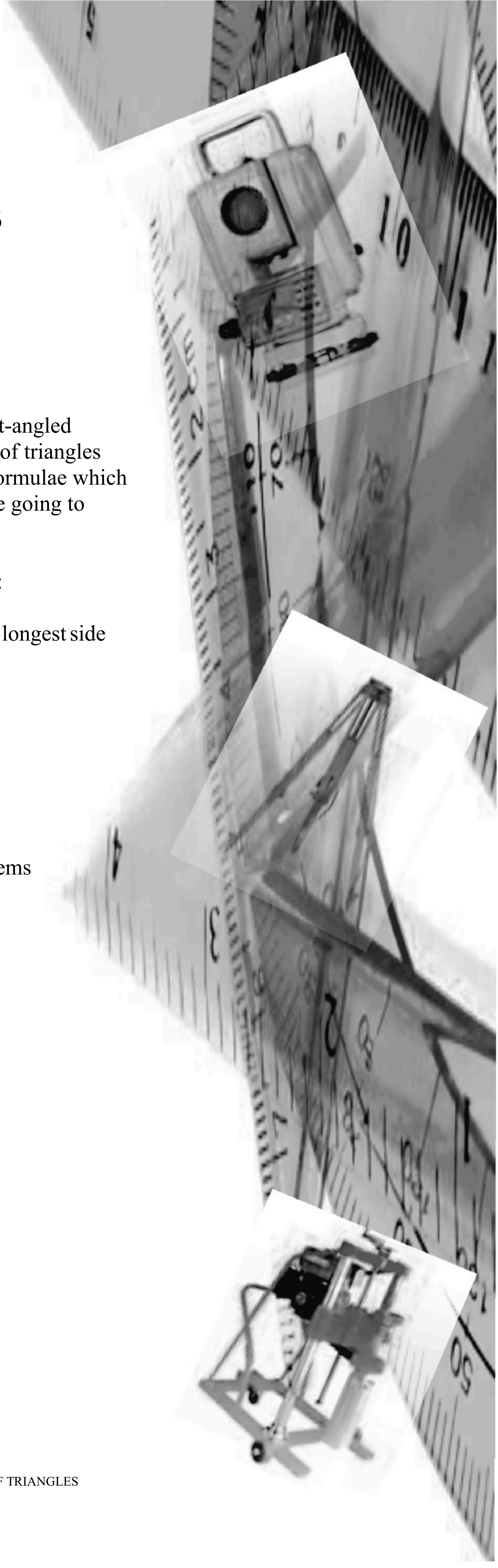
b) $2\sin\theta - \cos\theta = 0$

LESSON 5

Solution of Triangles



About this lesson

Since we started trigonometry, we have solved only right-angled triangles. It is possible however to find sides and angles of triangles which are not right-angled. This is done through using formulae which are known as cosine and sine rules. In this lesson you are going to learn about these rules.

You will need to remember properties of triangles, such as:

- the sum of angles of a triangle is 180°
- the side opposite the biggest angle in a triangle is the longest side
- the theorem of Pythagoras.

In this lesson you will:

- use the cosine rule to solve triangles
- use the area rule to find the areas of triangles
- use the sine rule to solve triangles
- apply area, sine and cosine rules to practical problems

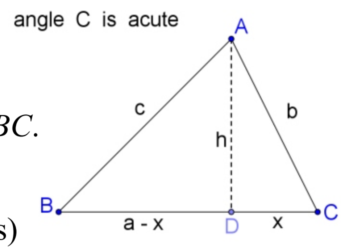
The Cosine Rule

The cosine rule is used to find the lengths of sides or sizes of angles of a triangle given either three sides or two sides and the angle between them.

The rule states:
In any triangle ABC, $c^2 = a^2 + b^2 - 2ab\cos C$

Let's see how this rule is derived. We shall look at case 1 where angle C in ΔABC is an acute angle, and case 2 where angle C in ΔABC is an obtuse angle.

Case 1



Look at the diagram:

Draw a perpendicular from vertex A to line BC.

In the ΔABC
$h^2 = b^2 - x^2$ (1) (By Pythagoras)

In the ΔABD
$h^2 = c^2 - (a - x)^2$ (2) (By Pythagoras)

Equating the right-hand sides of equations (1) and (2),
$b^2 - x^2 = c^2 - (a - x)^2$ (3) or

$b^2 - x^2 = c^2 - a^2 + 2ax + x^2$

$\therefore b^2 = c^2 + a^2 + 2ax$ (4)

From the ΔADC $\frac{x}{b} = \cos C$

Therefore, $x = b\cos C$

When we substitute for x, in equation (4), we get $b^2 = c^2 - a^2 + 2ab\cos C$

Rearranging the above equation gives the cosine rule where angle C is the angle between sides a and b.

cosine rule:
$c^2 = a^2 + b^2 - 2ab\cos C$

What happens case when angle C is obtuse?

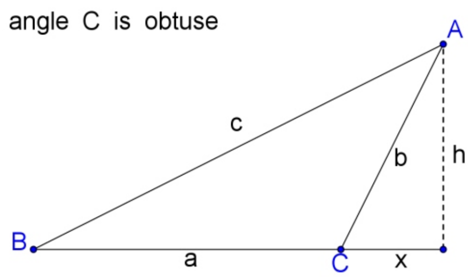
Case 2

Look at the diagram.

Draw AD perpendicular to BC produced.

In ΔADC
$h^2 = b^2 - x^2$ (5)

In ΔADC
$h^2 = c^2 - (a + x)^2$ (6)



Equating the right-hand sides of equations (5) and (6), we get the following,

$$b^2 - x^2 = c^2 - (a + x)^2 \qquad (7)$$

or $b^2 - x^2 = c^2 - a^2 - 2ax - x^2$

$$\therefore b^2 = c^2 - a^2 - 2ax \qquad (8)$$

From ΔADC, $\frac{x}{b} = \cos\left(180^\circ - C\right)$

$= -\cos C$

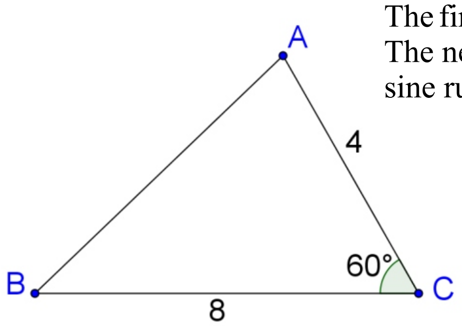
Therefore, $x = -b\cos C$

Substituting $x = -b\cos C$ into equation (8) gives $b^2 = c^2 - a^2 + 2ab\cos C$. Re-arranging the above equation gives the cosine rule $c^2 = a^2 + b^2 - 2ab\cos C$.

We use the cosine rule:

1. when given two sides of a triangle and the angle between them, to get the length of the third side. The third side will always be opposite the given or known angle.

Example 1



Given: ΔABC with side $b = 4$ and side $a = 8$ $\hat{C} = 60^\circ$, find side c (AB)

The first thing to do is to draw the triangle. The next thing will be to write down the sine rule and then substitute.

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$= 64 + 16 - 2.8.4.\cos 60^\circ$$
$$= 80 - 32$$
$$= 48$$
$$\therefore x = 6{,}9$$

2. When given all three sides of a triangle, we can use the cosine rule to find the internal angles of the triangle.

Example 2

Let us suppose that all the sides of the triangle in example 1 were given, i.e $a = 8, b = 4$ and $c = 6.9$. When naming a triangle ABC, we name the vertices (angles) of the triangle with capital letters A, B and C. Then name the sides opposite each vertex with the corresponding lower case letter: *a* opposite *A*, *b* opposite *B* and *c* opposite *C*.

We would be able to find angle C using the cosine rule, this way:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\therefore 6{,}9^2 = 8^2 + 4^2 - 2(8)(4)\cos C$$

$$\therefore 64\cos \hat{C} = 64 + 16 - 6{,}9^2$$

$$\therefore \cos \hat{C} = \frac{80 - 6{,}9^2}{64}$$

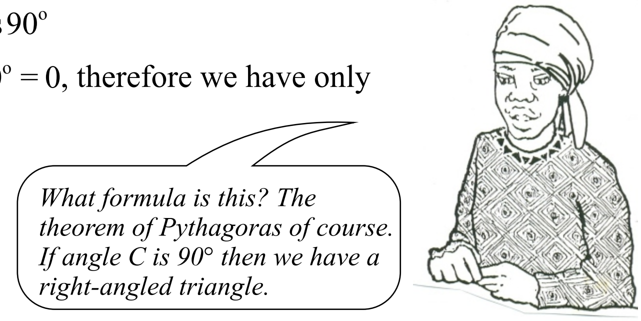
$$\therefore \hat{C} = 59{,}60^\circ$$

NOTE: When $\hat{C} = 90^\circ$, the formula is as follows:

$c^2 = a^2 + b^2 - 2ab\cos 90^\circ$

We know that $\cos 90^\circ = 0$, therefore we have only

$c^2 = a^2 + b^2$



Now try the following activity.

ACTIVITY 1

Calculate the side opposite the given angle in each of these triangles.

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1. $\hat{A} = 65^\circ,\ b = 3\ cm,\ c = 5\ cm$
2. $\hat{C} = 60^\circ,\ a = 6\ cm,\ b = 10\ cm$
3. $\hat{B} = 120^\circ,\ a = 8\ cm,\ c = 12\ cm$

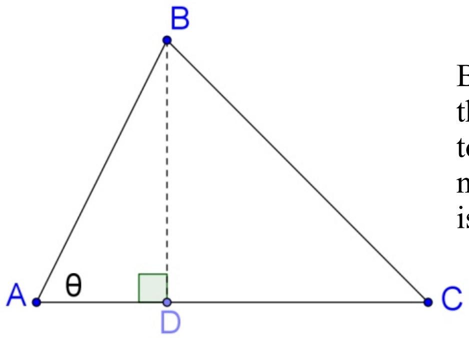
The Area Rule

In the past, we have always needed a perpendicular height of a triangle before we could calculate its area. Now, with the aid of trigonometry, we can work our way around this problem.

The formula used to calculate the area of a triangle is

$$Area\ \Delta = \frac{1}{2}.\ base\ .\perp height$$

Let's look at a sketch of any triangle ABC.



$$\text{Area } \Delta ABC = \frac{1}{2}.\text{ base}.\perp\text{height}$$

$$= \frac{1}{2}\ AC.BD$$

BD is the perpendicular height of this triangle, but we would prefer to have a method that does not need the perpendicular height. This is where we use trigonometry.

$$\sin\theta = \frac{BD}{AB}$$

$$\therefore BD = AB\sin\theta$$

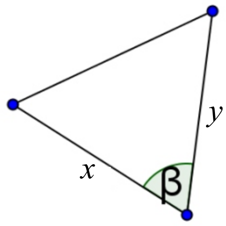
The area of the triangle can therefore be found using the formula:

$$\frac{1}{2}.AC.AB.\sin\theta$$

We now have a formula for any triangle where we are not given the height of the triangle.

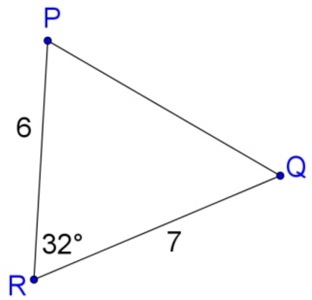
$$\text{Area } \Delta ABC = \frac{1}{2}.a.b.\sin C$$

So all we need to find the area is the lengths of two of the sides as well as the size of the angle in between them.



Example 3

Find the area of the triangle PQR below.



Solution

$$\text{Area } \Delta PQR = \frac{1}{2} p.q.\sin R$$

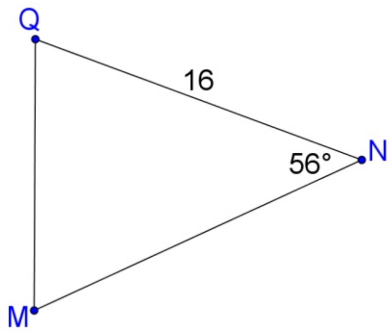
$$= \frac{1}{2}(7)(6)(\sin 32^{\circ})$$

$$= 9{,}27 \text{units}^2$$

We can also use the Area Rule when we are given the area of a triangle and asked to find the dimensions of the triangle.

Example 4

If we are given that the area of ΔMNQ is $70\,cm^2$, calculate the lengths of q and n.



Solution

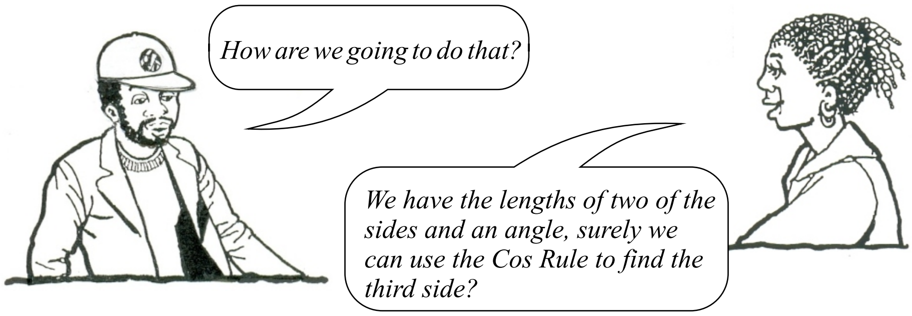
$\text{Area } \Delta MNQ = \frac{1}{2}.16.q.\sin 56^{\circ}$ Using the formula for the Area of a Triangle

$\therefore 70 = 8q\sin 56^{\circ}$ Substituting the given area

$$q = \frac{70}{8\sin 56}$$

$$= 10{,}55\,cm$$

We have now found the length of q. We still need to find the length of n.



Correct, let's calculate the length of n.

$$n^2 = 10.55^2 + 16^2 - 2\,(10.55)(16)\cos 56^\circ$$
$$= 367.3025 - 337.6\cos 56^\circ$$
$$= 178{,}52$$
$$\therefore n = 13{,}36\,cm$$

Now try the activity on your own.

ACTIVITY 2

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The area of $\Delta XYZ = 30\,cm^2$. $\hat{Y} = 112^\circ$ and $x = 5{,}2cm$.
Determine the lengths of y and z.

The sine rule

The sine rule is used to solve triangles in which two angles and any side are given. Although the cosine rule assists us in finding the length of a side when two sides and an angle are given, if the given angle is not between the two given sides then the cosine rule is difficult to apply. The sine rule can deal with that situation and we can also use the sine rule when two sides and an angle opposite one of them is given. We can write this rule as follows:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

In the same way as we did with the cosine rule, we can show that the equalities written above are true for all triangles.

This proof uses the formula for area developed earlier.

The area of $\Delta ABC = \frac{1}{2}a.b\,\sin C$. This can of course be written in terms of any angle. That is, the area of the same triangle could also be given by $\frac{1}{2}b.c.\sin A$, as well as $\frac{1}{2}a.c.\sin B$. These all represent the area of the same triangle therefore

$$\frac{1}{2}a.b.\sin C = \frac{1}{2}b.c.\sin A = \frac{1}{2}a.c.\sin B$$

If we now divide through by $\frac{1}{2}a.bc$ we are left with the sine rule.

$$\frac{\frac{1}{2}a.b.\sin C}{\frac{1}{2}a.b.c} = \frac{\frac{1}{2}b.c.\sin A}{\frac{1}{2}a.b.c} = \frac{\frac{1}{2}a.c.\sin B}{\frac{1}{2}a.b.c}$$

$$\therefore \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

This rule helps us to solve triangles where either two angles and a side are given, or when two sides and an angle opposite one of the sides is given.

In all problems it is advisable to draw a rough sketch of the triangle described.

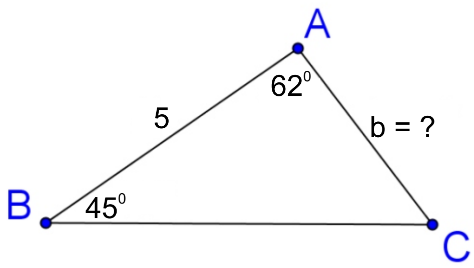
Example 5
In ΔABC , $\hat{B} = 45^o$, $\hat{A} = 62^o$, $c = 5\,cm$, find b.

Solution
Start by drawing a rough sketch of the triangle.

Since finding angle C is straight forward, let's start with that.

$\hat{C} = 180^o - 45^o - 62^o = 73^o$

Now we apply the sine rule.



$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

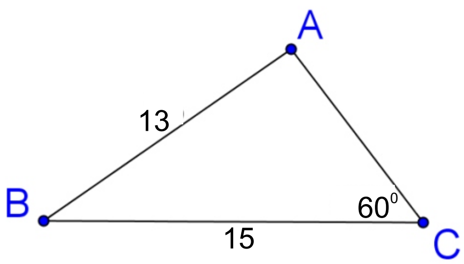
$$\frac{b}{\sin 45^o} = \frac{5}{\sin 73^o}$$

$$\therefore b = \frac{5\sin 45^o}{\sin 73^o}$$

$$= 3{,}70\,cm$$

Example 6
Solve the ΔABC completely in which $a = 15\,cm$, $c = 13\,cm$ and $\hat{C} = 60^o$

Solution
Again, we start with a rough sketch.



Use the sine rule since the given angle in not between the two given sides.

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\therefore \frac{\sin A}{15} = \frac{\sin 60}{13}$$

$$\therefore \sin A = \frac{15\sin 60}{13}$$

Now we have to use our knowledge of equations to solve for $\hat{A}$. Remember that when solving an equation using sine, we need to consider two options. The angle we calculate and 180° minus that angle.

$\therefore \hat{A} = 87,80^{\circ}$ or $\hat{A} = 92,20^{\circ}$

ambiguous:
something that has a double meaning

Since neither of these angles exceed 180°(the maximum possible in a triangle) when added to the already existing angle of 60°, neither can be discarded and we need to follow each possibility through to the end of the problem. We refer to this as the 'Ambiguous Case'.

So, let's continue with both angles.

$\therefore \hat{A} = 87,80^{\circ}$ or $\hat{A} = 92,20^{\circ}$

$\therefore \hat{B} = 32,20^{\circ}$ $\therefore \hat{B} = 27,80^{\circ}$

$\therefore \dfrac{b}{\sin 32,20^{\circ}} = \dfrac{13}{\sin 60^{\circ}}$ $\therefore \dfrac{b}{\sin 27,80^{\circ}} = \dfrac{13}{\sin 60^{\circ}}$

$\therefore b = \dfrac{13 \sin 32,20^{\circ}}{\sin 60^{\circ}}$ $\therefore b = \dfrac{13 \sin 27,80^{\circ}}{\sin 60^{\circ}}$

$= 8\,cm$ $= 7\,cm$

Both sets of answers are valid.
Try this activity now.

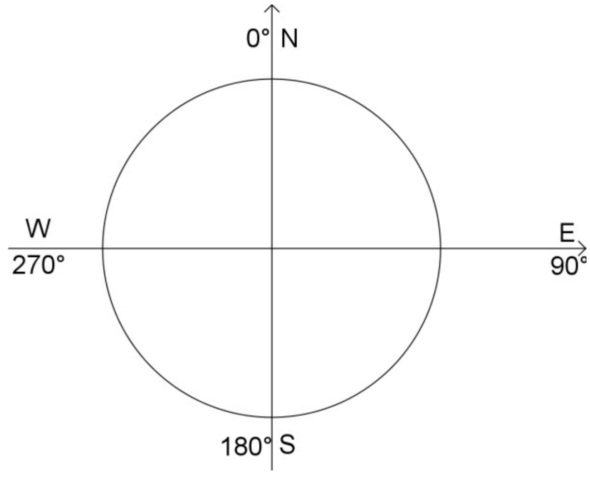
ACTIVITY 3

Solve the following triangles completely (i.e., find all the missing dimensions).

1. $\hat{A} = 115^{\circ},\ a = 65\,cm,\ b = 32\,cm$
2. $\hat{A} = 77^{\circ},\ \hat{C} = 41^{\circ},\ b = 4.3\,cm$
3. $\hat{B} = 42^{\circ},\ c = 24\,cm,\ b = 18\,cm$

ANSWERS ON PAGE 64

Applications of area, sine and cosine rules



The four directions north, east, south and west, can be drawn as we draw the x and y –axes. The north line will be the positive y –axis. This line, north, is taken as 0^{o}. From this line, angles are measured clockwise. Look at the diagram on the previous page. These angles are called bearings.

A bearing is often given using three numbers. For example, $36^{o}$is written as 036^{o}. If the bearing is not given, then you will be told whether the direction is west, east, south-west, etc.

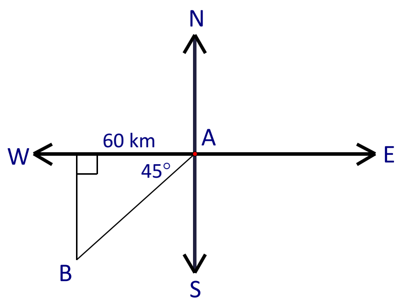
Sine and cosine rules have a wide application such as in navigation and surveying. The following are a few of the examples you may come across in everyday life.

Example 7

A ship sails due west for 60 km. It then changes course and moves on a bearing of 180^{o} until it is south-west of its starting point. How far is it then from its starting point?

Solution

The adjacent drawing shows that the movement on a bearing of 180^{o} gives an angle of 90^{o} inside the triangle. The angle given as south-west is on a bearing of $180^{o} + 45^{o} = 225^{o}$ from A.



This gives an angle of 45^{o} inside the triangle.
Using trigonometry, we can get the length of AB as follows:

$$\frac{AB}{\sin 90^{o}} = \frac{60}{\sin 45^{o}}$$

$$\therefore AB = \frac{60(1)}{\sin 45^{o}}$$

$$= 84{,}85\,km$$

The ship will then be about 85 km from its starting position.

Example 8

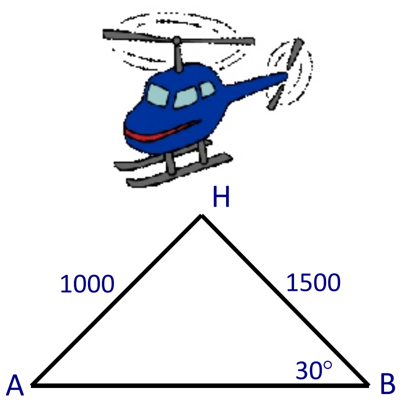
A helicopter hovers at a point in the same vertical plane as two points A and B on the ground. It is 1000 m and 1500 m from A and B respectively. The angle of elevation from B is 30°. Find the distance AB.

Solution

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\therefore \sin A = \frac{1500.\sin 30^{o}}{1\ 000}$$

$$\therefore A = 48{,}59^{o} \quad or \quad A = 131{,}41^{o}$$



$$\therefore H = 101{,}41^{o}$$
$$\therefore \frac{AB}{\sin 101{,}41^{o}} = \frac{1000}{\sin 30^{o}}$$
$$\therefore AB = \frac{1000.\sin 101{,}41^{o}}{\sin 30}$$
$$= 1960{,}47\,m$$

or

$$\therefore H = 18{,}59^{o}$$
$$\therefore \frac{AB}{\sin 18{,}59^{o}} = \frac{1000}{\sin 30^{o}}$$
$$\therefore AB = \frac{1000.\sin 18{,}59^{o}}{\sin 30}$$
$$= 637{,}59\,m$$

ACTIVITY 4

A and B are observation points 5 km apart. B is due east of A. The bearing of an opponent's position is 030^{o} from A and 015^{o} from B. How far from A is the position of the opponent?

ANSWERS ON PAGE 65

Summary

You should be able to state or write the area, sine and cosine rules. You should also be able to recognise when you can use each of the two rules.

They are used to solve problems related to finding areas, sides and angles in triangles that need not be right-angled. You can use the sine rule if you have been given two angles and any side, or two sides and an angle opposite one of the sides. You can use the cosine rule if you have been given two sides and an angle between them or three sides.

You have seen that the sine and cosine rules can be applied in navigation, surveying, etc. There are more applications that you will come across if you work with triangles.

CHECKLIST

Are you able to:

- write down the cosine rule, area rule and the sine rule
- use the cosine rule to solve a triangle
- use the sine rule to solve a triangle
- use the area rule

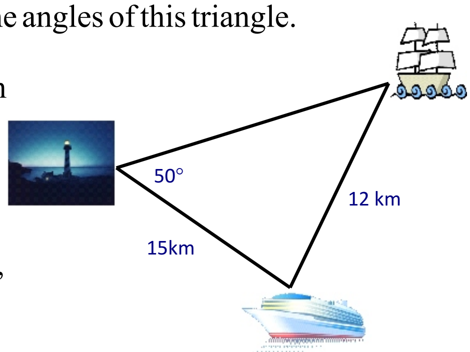
Check yourself in the following exercise.

SELF-CHECK EXERCISE

1. In $\Delta ABC, \hat{A} = 35^{o}, \hat{B} = 42, \; b = 15.m$ find a.

2. $a = 5, b = 2, c = 4$ Find all the angles of this triangle.

3. Two ships leave a harbour in directions which diverge from one another by 50°. After an hour the ships are 12 km apart. If the faster ship is travelling at 15 km/h, what is the speed of the slower one?



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Feedback to Activities

Lesson 1

Activity 1

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4(3)(1)}}{6}$$

$$= \frac{-2 \pm \sqrt{-8}}{6}$$

Again the square root of a negative number is not a real number. So there is no **real** answer for this question.

Activity 2

a) $x = 2 \ \ or \ \ x = 3$

$\therefore x - 2 = 0 \ \ or \ \ x - 3 = 0$

$\therefore (x-2)(x-3) = 0$

$\therefore x^2 - 5x + 6 = 0$

b) $x = 2 \ \ or \ \ x = -3$

$\therefore x - 2 = 0 \ \ or \ \ x + 3 = 0$

$\therefore (x-2)(x+3) = 0$

$\therefore x^2 + x - 6 = 0$

c) $x = -2 \ \ or \ \ x = 3$

$\therefore x + 2 = 0 \ \ or \ \ x - 3 = 0$

$\therefore (x+2)(x-3) = 0$

$\therefore x^2 - x - 6 = 0$

d) $x = -2 \ \ or \ \ x = -3$

$\therefore x + 2 = 0 \ \ or \ \ x + 3 = 0$

$\therefore (x+2)(x+3) = 0$

$\therefore x + 5x + 6 = 0$

e) $x = 4 \ \ or \ \ x = 4$

$\therefore x - 4 = 0 \ \ or \ \ x - 4 = 0$

$\therefore (x-4)(x-4) = 0$

$\therefore x^2 - 8x + 16 = 0$

Did you see that the answer to e) can also be written as $(x-4)^2 = 0$

f) $x = -\frac{3}{2}$ *or* $x = -\frac{1}{8}$

$\therefore 2x = -3$ *or* $8x = -1$

$\therefore 2x + 3 = 0$ *or* $8x + 1 = 0$

$\therefore (2x + 3)(8x + 1) = 0$

$\therefore 16x^2 + 26x + 3 = 0$

Activity 3

These equations usually look more difficult than they are.

a) Let's start by getting a common denominator on both sides

$$\frac{x\sqrt{3}\left(5 - x\sqrt{3}\right)}{\left(2\sqrt{3} - x\right)\left(5 - x\sqrt{3}\right)} = \frac{2x\left(2\sqrt{3} - x\right)}{\left(2\sqrt{3} - x\right)\left(5 - x\sqrt{3}\right)}$$

At this point you can get rid of the denominator easily by multiplying both sides of the equation by the lowest common denominator.

$$x\sqrt{3}\left(5 - x\sqrt{3}\right) = 2x\left(2\sqrt{3} - x\right)$$

$$5x\sqrt{3} - 3x^2 = 4x\sqrt{3} - 2x^2$$

$$-x^2 + 5x\sqrt{3} - 4x\sqrt{3} = 0$$

$$-x^2 + x\sqrt{3} = 0$$

$$x^2 - x\sqrt{3} = 0$$

$$x\left(x - \sqrt{3}\right) = 0$$

$$\therefore x = 0 \quad or \quad x = \sqrt{3}$$

b) When choosing our lowest common denominator, we see that we can make things easier by first doing a switch-round on the right-hand side

$$\frac{y+2}{y+1} = \frac{y-2}{1-y} + \frac{4}{1-y}$$

$$\frac{y+2}{y+1} = \frac{y-2+4}{1-y}$$

$$\frac{y+2}{y+1} = \frac{y+2}{1-y}$$

Now we make a common denominator on both sides.

$$\frac{(y+2)(1-y)}{(y+1)(1-y)} = \frac{(y+2)(y+1)}{(y+1)(1-y)}$$

$$(y+2)(1-y) = (y+2)(y+1)$$

$$y - y^2 + 2 - 2y = y^2 + 3y + 2$$

$$-2y^2 - 4y = 0$$

$$2y^2 + 4y = 0$$

$$2y(y+2) = 0$$

$$\therefore y = 0 \quad or \quad y = -2$$

Activity 4

Let the number of cows at farm A be *x*.

Therefore, the number of cows at farm B will be $x - 60$

Annual yield per cow at farm A is $\frac{820}{x}$

Annual yield per cow at farm B is $\frac{1050}{x-60}$

$$\frac{1050}{x-60} - \frac{820}{x} = 1$$

$$\frac{1050(x) - 820(x-60)}{x(x-60)} = \frac{x(x-60)}{x(x-60)}$$

$$1050x - 820x + 49200 = x^2 - 60x$$

$$-x^2 + 290x + 49200 = 0$$

$$x^2 - 290x - 49200 = 0$$

Although we could find factors to solve this quadratic equation, it might just be easier to use the formula.

$$x = \frac{-(-290) \pm \sqrt{(-290)^2 - 4(1)(-49200)}}{2(1)}$$

$$= \frac{290 \pm \sqrt{280900}}{2}$$

$$= \frac{290 \pm 530}{2}$$

$$\therefore x = 410 \quad or \quad x = -120$$

But you can't have a negative number of cows. So the number of cows at farm A must be **410** and the number of cows at farm B must be $410 - 60 = 350$.

Now you can use these values to calculate the annual milk production per cow at each farm.

Annual yield per cow on farm A is $\frac{820}{410} = 2$ tons per year.

Annual yield per cow on farm B is $\frac{1050}{350} = 3$ tons per year.

Lesson 2

Activity 1

You can always express all the trigonometric ratios in terms of sin and cos and then cancel out.

1. Prove: $\tan A.\cos A = \sin A$

   Proof: $LHS = \tan.A \cos A$

$$= \frac{\sin A}{\cos A} \times \frac{\cos A}{1}$$
$$= \sin A$$
$$= RHS$$

2. Prove: $1-(\sin A - \cos A)(\sin A + \cos A) = 2\sin^2 A$

   Proof: $LHS = 1-(\sin A - \cos A)(\sin A + \cos A)$

$= 1-(\sin^2 A - \cos^2 A)$ (multiply the brackets out)

$= 1 - \sin^2 A + \cos^2 A$

$= \cos^2 A + \cos^2 A$ (using the squared identity)

$= 2\cos^2 A$

$= RHS$

Activity 2

1. a)

$$LHS = \sin(90^\circ - \theta)$$
$$= \sin 90^\circ \cos\theta - \cos 90^\circ \sin\theta$$
$$= 1\times\cos\theta - 0\times\sin\theta$$
$$= \cos\theta$$
$$= RHS$$

b)

$$LHS = \cos(90^\circ + \theta)$$
$$= \cos 90^\circ \cos\theta - \sin 90^\circ \sin\theta$$
$$= 0\times\cos\theta - 1\times\sin\theta$$
$$= -\sin\theta$$
$$= RHS$$

2. a) $\cos A \sin B + \sin A \cos B = \sin A \cos B + \cos A \sin B$

$$= \sin(A + B)$$

b) Using the identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\cos 50^{\circ} \cos 40^{\circ} - \sin 50^{\circ} \sin 40^{\circ} = \cos(50^{\circ} + 40^{\circ})$$

$$= \cos 90^{\circ}$$

$$= 0$$

c) Taking out the common factor -1 we can change the expression to look like this:

$$-(\cos 3x \cos x + \sin 3x \sin x)$$

We will then use the identity to express it as follows:

$$-(\cos(3x - x))$$

$$= -\cos 2x$$

Activity 3

1. $\cos 2\alpha = \cos(\alpha + \alpha)$

$$= \cos\alpha \cos\alpha - \sin\alpha \sin\alpha$$

$$= \cos^2\alpha - \sin^2\alpha$$

2. $\sin 2\alpha = \sin(\alpha + \alpha)$

$$= \sin\alpha \cos\alpha + \sin\alpha \cos\alpha$$

$$= 2\sin\alpha \cos\alpha$$

Activity 4

1. $LHS = \dfrac{\sin 2A}{1 + \cos 2A}$

$$= \frac{2\sin A \cos A}{1 + \cos^2 A - \sin^2 A}$$

$$= \frac{2\sin A \cos A}{1 - \sin^2 A + \cos^2 A}$$

$$= \frac{2\sin A \cos A}{\cos^2 A + \cos^2 A}$$

$$= \frac{2\sin A \cos A}{2\cos^2 A}$$

$$= \frac{\sin A}{\cos A}$$

$$= \tan A$$

> ***Compound Angle Identity***
>
> $\sin 3x = \sin(2x + x)$
> $\quad = \sin 2x \cos x + \cos 2x \sin x$

> ***Pythagorean Identity***
>
> $1 - \sin^2 x = \cos^2 x$

2. $$LHS = \frac{\sin 3x + \sin x}{1 + \cos 2x}$$

$$= \frac{\sin 2x \cos x + \cos 2x \sin x + \sin x}{1 + \cos^2 x - \sin^2 x}$$

$$= \frac{2\sin x \cos x \cos x + \cos 2x \sin x + \sin x}{\left(1 - \sin^2 x\right) + \cos^2 x}$$

$$= \frac{\sin x\left(2\cos^2 x + \cos 2x + 1\right)}{\cos^2 x + \cos^2 x}$$

$$= \frac{\sin x\left(2\cos^2 x + \cos^2 x - \sin^2 x + 1\right)}{2\cos^2 x}$$

$$= \frac{\sin x\left(3\cos^2 x + 1 - \sin^2 x\right)}{2\cos^2 x}$$

$$= \frac{\sin x\left(3\cos^2 x + \cos^2 x\right)}{2\cos^2 x}$$

$$= \frac{4\sin x \cos^2 x}{2\cos^2 x}$$

$$= 2\sin A$$

$$= \text{RHS}$$

Lesson 3

Acitivity 1

$$LHS = \frac{\sin 3\alpha}{\sin \alpha} - \frac{\cos 3\alpha}{\cos \alpha}$$

$$= \frac{\sin 3\alpha \cos \alpha - \cos 3\alpha \sin \alpha}{\sin \alpha \cos \alpha}$$

$$= \frac{\sin(3\alpha - \alpha)}{\frac{1}{2}\sin 2\alpha}$$

$$= \frac{2\sin 2\alpha}{\sin 2\alpha}$$

$$= 2$$

$$= \text{RHS}$$

Activity 2

1.	$x = 35^{\circ}$	6.	$\beta = 60^{\circ}$
2.	$x = 30^{\circ}$	7.	$x = 60^{\circ}$
3.	$y = -60^{\circ}$	8.	$A = 45^{\circ}$
4.	$\alpha = 45^{\circ}$	9.	$\beta = -30^{\circ}$
5.	$y = 180^{\circ}$	10.	$x = 45^{\circ}$

Activity 3

1. $-3\sin x + 2 = 0$

   $\therefore \sin x = \frac{2}{3}$

   $\therefore x = 41{,}81^{\circ} + k.360^{\circ}$ or $x = 180^{\circ} - 41{,}81^{\circ} + k.360^{\circ}$

   $= 138{,}19^{\circ} + k.360^{\circ}$ where $k \in Z$

2. $3\cos(x - 20) + 1 = 0$

   The important thing here is to remember that first we have to solve for the $(x - 20^{\circ})$ angle and then only do we solve for *x*.

   $\therefore \cos(x - 20^{\circ}) = -\frac{1}{3}$

   $\therefore x - 20^{\circ} = 109{,}47^{\circ} + k.360^{\circ}$ or $\therefore x - 20^{\circ} = -109{,}47^{\circ} + k.360^{\circ}$

   $\therefore x = 129{,}47^{\circ} + k.360^{\circ}$ or $x = -89{,}47^{\circ} + k.360^{\circ}$ where $k \in Z$

3. $\tan 2x = 3$

   We cannot clear the 2 yet as it is part of the angle. First we must solve for the angle and then we can solve for *x*.

   $\therefore 2x = 71{,}57^{\circ} + k.180^{\circ}$

   $\therefore x = 35{,}78^{\circ} + k.90^{\circ}$ where $k \in Z$

Activity 4

1. When you come across questions that have $\sin x \cos x$, you should consider whether you can use the identity $\sin 2x = 2\sin x \cos x$

   $\sin x \cos x = \frac{\sqrt{3}}{4}$

   $\therefore 2\sin x \cos x = \frac{\sqrt{3}}{2}$ (multiply both sides by 2 to make the left hand side familiar)

   $\therefore \sin 2x = \frac{\sqrt{3}}{2}$

   $\therefore 2x = 60^{\circ} + k.360^{\circ}$ or $2x = 180^{\circ} - 60^{\circ} + k.360^{\circ}$

   $\therefore x = 30^{\circ} + k.180^{\circ}$ or $x = 60^{\circ} + k.360^{\circ}$ where $k \in Z$

2. $\sin x = \sin 2x$

   $\therefore \sin x = 2\sin x \cos x$

   $\therefore \sin x - 2\sin x \cos x = 0$

   $\therefore \sin x(1 - 2\cos x) = 0$ Now we have a product of two factors equal to 0

Each factor could be equal to zero, and so we now have two separate equations to solve.

$\therefore \sin x = 0$ or $\cos x = \frac{1}{2}$

$\therefore x = 0^\circ + k.360^\circ$ or $x = 180^\circ + k.360^\circ$ or $x = 60^\circ + k.360^\circ$
or $x = -60^\circ + k.360^\circ$ where $k \in Z$

3. $\cos 2x = \cos x$
$\therefore \cos^2 x - \sin^2 x - \cos x = 0$
$\therefore \cos^2 x - (1 - \cos^2 x) - \cos x = 0$
$\therefore \cos^2 x - 1 + \cos^2 x - \cos x = 0$
$\therefore 2\cos^2 x - \cos x - 1 = 0$

Can you see that this is a quadratic trinomial? If not, try making $\cos x = k$ and looking again.

$\therefore (2\cos x + 1)(\cos x - 1) = 0$

$\therefore \cos x = -\frac{1}{2}$ or $\cos x = 1$

$\therefore x = 120^\circ + k.360^\circ$ or $x = -120^\circ + k.360^\circ$ or $x = 0^\circ + k.360^\circ$
where $k \in Z$

Activity 5

1. $\cos(2x + 15^\circ) = \sin x$

$\therefore \cos(2x + 15^\circ) = \cos(90^\circ - x)$

$\therefore 2x + 15^\circ = 90^\circ - x + k.360^\circ$ or $2x + 15^\circ = -(90^\circ - x) + k.360^\circ$
$\therefore 3x = 75^\circ + k.360$ or $x = -105^\circ + k.360^\circ$
$\therefore x = 25^\circ + k.120^\circ$ where $k \in Z$

2. $\sin 2x \cos 20 + \cos 2x \sin 200 = \cos 2x$

$\therefore \sin 2x \cos 20^\circ + \cos 2x(-\sin 20^\circ) = \cos 2x$

$\therefore \sin 2x \cos 20 - \cos 2x \sin 20 = \cos 2x$

$\therefore \sin(2x - 20^\circ) = \cos 2x$

$\therefore \sin(2x - 20^\circ) = \sin(90^\circ - 2x)$

$\therefore 2x - 20^\circ = 90^\circ - 2x + k360$ or $2x - 20^\circ = 180^\circ - (90^\circ - 2x) + k.360^\circ$

$\therefore 4x = 110^\circ + k.360^\circ$ or $2x - 20^\circ = 180^\circ - 90^\circ + 2x + k.360^\circ$

$\therefore x = 27{,}5^\circ + k.90^\circ$ or No solution. Where $k \in Z$

Lesson 4

Activity 1

$2\sin\theta - \sqrt{3} = 0$

$\therefore \sin\theta = \dfrac{\sqrt{3}}{2}$

$\therefore \theta = 60^\circ + k.360^\circ$ or $\theta = 180^\circ - 60^\circ + k.360^\circ$
$= 120^\circ + k.360^\circ$ where $k \in Z$

Activity 2

1. $3\tan\theta + 1 = 0$ for $\theta \in [-360^\circ; 180^\circ]$

   $\tan\theta = -\dfrac{1}{3}$

   $\therefore \theta = 18{,}43^\circ + k.180^\circ$ where $k \in Z$

   Now we need to find all the solutions that fall within the limit.
   $\therefore \theta = 18{,}43^\circ; -161{,}57^\circ; -341{,}57^\circ$

2. $\dfrac{\sin 2\theta}{\cos\theta} - 1 = 0$ for $\theta \in [-180^\circ, 360^\circ]$

   $\therefore \dfrac{2\sin\theta\cos\theta}{\cos\theta} - 1 = 0$ (using the double angle identity for sine)

   $\therefore 2\sin\theta = 1$

   $\therefore \sin\theta = \dfrac{1}{2}$

   $\therefore \theta = 30^\circ + k.360^\circ$ or $\theta = 150^\circ + k.360^\circ$ where $k \in Z$

   In attempting to solve the equation we use identities that will simplify the equation until the angle theta is expressed in one function only. It is only then that we find the angle required.
   $0 = 30^\circ ; 150^\circ$

3. $\sin 2\theta + 2\sin 2\theta \cos 2\theta = 0$ for $\theta \in [-360^\circ ; 0^\circ]$
   The above equation needs factorisation.

   $\sin 2\theta\,(1 + \cos 2\theta) = 0$

   $\therefore \sin 2\theta = 0$ or $\cos 2\theta = -1$
   $\therefore 2\theta = 0^\circ + k.360^\circ$ or $2\theta = 180^\circ + k.360^\circ$ or $2\theta = \pm 180^\circ + k.360^\circ$
   $\therefore \theta = 0^\circ + k.180^\circ$ or $\theta = 90^\circ + k.180^\circ$ or $\theta = \pm 90^\circ + k.180^\circ$
   where $k \in Z$

   $\theta = -180^\circ, -360^\circ, -90^\circ, -270^\circ$

Activity 3

1. $2\sin x\tan x - \sin x - 2\tan x + 1 = 0$

$\therefore \sin x(2\tan x - 1) - (2\tan x - 1) = 0$

$\therefore (2\tan x - 1)(\sin x - 1) = 0$

$\therefore \tan x = \frac{1}{2}$ or $\sin x = 1$

$\therefore x = 26{,}57^\circ + k.180^\circ$ or $x = 90^\circ + k.360^\circ$ where $k \in Z$

Although this equation does not involve fractions with an unknown in the denominator, it still results in an invalid answer. This is because the function $y = \tan x$ is undefined for $x = 90^\circ$ and this answer therefore needs to be discarded leaving us with only one option.

$\therefore x = 26{,}57^\circ + k.180^\circ$ where $k \in Z$

2. $\sin^2 x - 10\sin x\cos x + 21\cos^2 x = 0$

$\therefore (\sin x - 7\cos x)(\sin x - 3\cos x) = 0$

$\therefore \sin x = 7\cos x$ or $\sin x = 3\cos x$

$\therefore \tan x = 7$ or $\tan x = 3$

$\therefore x = 81{,}87^\circ + k.180^\circ$ or $x = 71{,}57^\circ + k.180^\circ$ where $k \in Z$

3. $\sin(x - 90^\circ) - \sin 90^\circ = \sin(x + 90^\circ)$

$\therefore \sin x\cos 90^\circ - \cos x\sin 90^\circ - 1 = \sin x\cos 90^\circ + \cos x\sin 90^\circ$

$\therefore \sin x(0) - \cos x(1) - 1 = \sin x(0) + \cos x(1)$

$\therefore -2\cos x = 1$

$\therefore \cos x = -\frac{1}{2}$

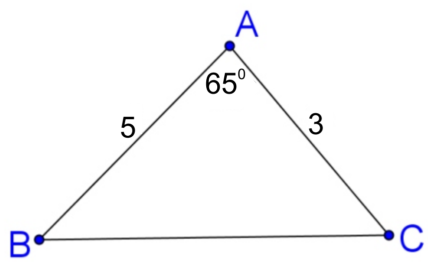
$\therefore = \pm 120^\circ + k.360^\circ$

Lesson 5

Activity 1

The best thing to do first is to draw a rough sketch of the triangle in question. In all the cases we can use the cosine rule to get the length of the unknown side.

1.



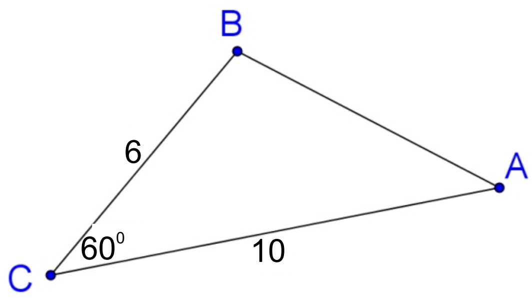
$a^2 = 5^2 + 3^2 - 2(3)(5)\cos 65^\circ$

$= 34 - 30\cos 65^\circ$

$= 21{,}32$

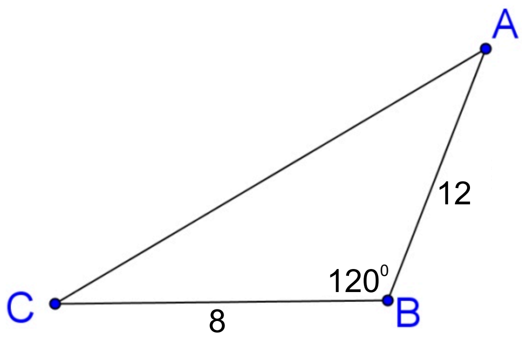
$\therefore a = 4{,}62cm$

2.



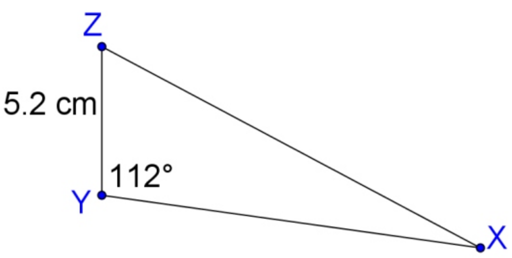
$$c^2 = 6^2 + 10^2 - 2(6)(10)\cos 60^{\circ}$$
$$= 136 - 120\cos 60^{\circ}$$
$$= 76$$
$$\therefore c = 8,72cm$$

3.



$$b^2 = 8^2 + 12^2 - 2(8)(12)\cos 120^{\circ}$$
$$= 208 - 192\cos 120^{\circ}$$
$$= 304$$
$$\therefore b = 17,44cm$$

Activity 2



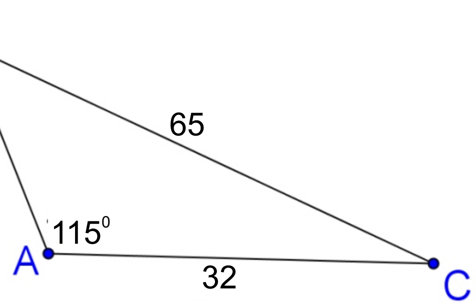
$$\text{Area } \Delta XYZ = \frac{1}{2}.5,2.z.\sin 112^{\circ}$$
$$\therefore 30 = 2,6z \sin 112^{\circ}$$
$$z = \frac{30}{2,6\sin 112^{\circ}}$$
$$= 12,44\,cm$$

$$y^2 = 5,2^2 + 12,44^2 - 2(5,2)(12,44)\cos 112^{\circ}$$
$$= 230,26$$
$$\therefore y = 15,17\,cm$$

Activity 3

In all the problems, draw a rough diagram of the triangle in question.

1.



$$\frac{\sin B}{32} = \frac{\sin 115^{\circ}}{65}$$

$$\therefore \sin B = \frac{32 \sin 115^{\circ}}{65}$$

$$\therefore B = 26{,}50^{\circ} \quad or \quad B = 153{,}50^{\circ}$$

In this case, when $153{,}50^{\circ}$ is added to the already existing angle of 115° it exceeds 180°. This makes the second option in this case invalid and so we discard that option and continue with the other angle.

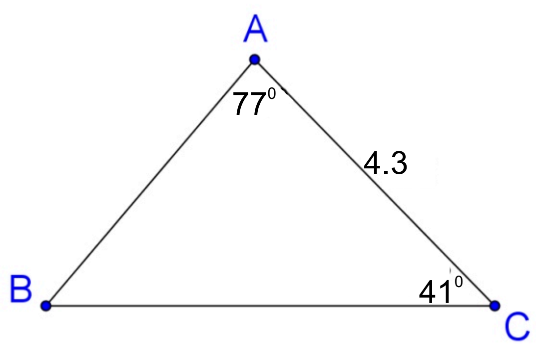
$$\therefore \hat{C} = 38{,}50^{\circ}$$

$$\frac{c}{\sin 38{,}50^{\circ}} = \frac{32}{\sin 26{,}50^{\circ}}$$

$$\therefore c = \frac{32 \sin 38{,}50^{\circ}}{\sin 26{,}50^{\circ}}$$

$$= 44{,}64\, cm$$

2.



$$\hat{B} = 62^{\circ}$$

$$\frac{a}{\sin 77^{\circ}} = \frac{4{,}3}{\sin 62^{\circ}}$$

$$\therefore a = \frac{4{,}3 \sin 77^{\circ}}{\sin 62^{\circ}}$$

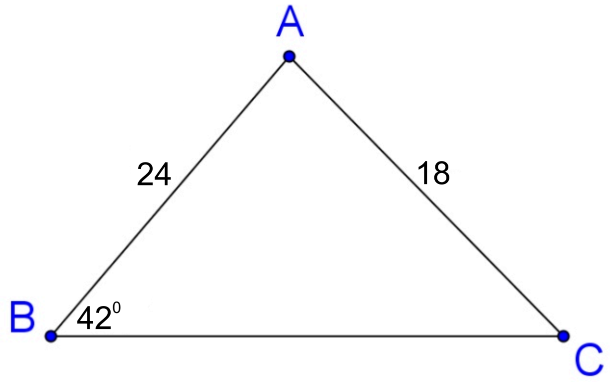
$$= 4{,}75\, cm$$

$$\frac{c}{\sin 41^{\circ}} = \frac{4{,}3}{\sin 62^{\circ}}$$

$$\therefore c = \frac{4{,}3 \sin 41^{\circ}}{\sin 62^{\circ}}$$

$$= 3{,}20\, cm$$

3.



$$\frac{\sin C}{24} = \frac{\sin 42^\circ}{18}$$

$$\therefore \sin C = \frac{24\sin 42^\circ}{18}$$

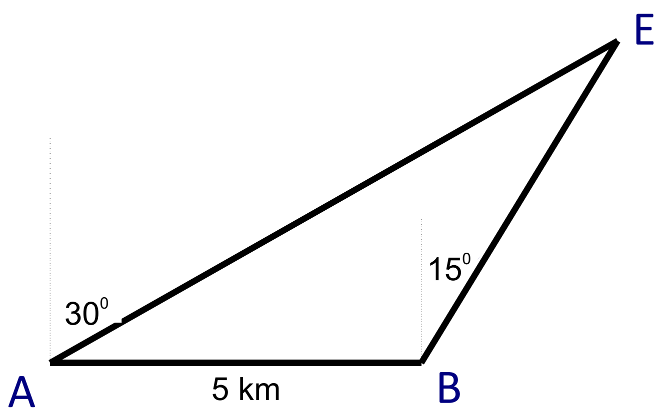
$$\therefore C = 63{,}15^\circ \qquad or \qquad C = 116{,}85^\circ$$

Both options for C are valid therefore this is an ambiguous case.
Both options must be explored.

$\therefore \hat{C} = 63{,}15^\circ$ or	$\hat{C} = 116{,}85^\circ$
$\therefore \hat{A} = 74{,}85^\circ$	$\therefore \hat{A} = 21{,}15^\circ$
$\therefore \frac{a}{\sin 74{,}85^\circ} = \frac{18}{\sin 42^\circ}$	$\therefore \frac{a}{\sin 21{,}15^\circ} = \frac{18}{\sin 42^\circ}$
$\therefore b = \frac{18\sin 74{,}85^\circ}{\sin 42^\circ}$	$\therefore a = \frac{18\sin 21{,}15^\circ}{\sin 42^\circ}$
$= 25{,}97\,cm$	$= 9{,}71\,cm$

Activity 4

The first thing to do in all problems of this type is to draw a rough sketch of the problem. This helps to make things clearer.



From the diagram it is clear that in triangle ABE angle $\hat{A} = 60^{o}$ and $\hat{B} = 105^{o}$

Then, $\hat{E} = 15^{o}$

.

$$\frac{EA}{\sin 105^{o}} = \frac{5}{\sin 15^{o}}$$

$$\therefore EA = \frac{5\sin 105^{o}}{\sin 15^{o}}$$

$$= 18{,}66$$

The distance from the opponent to observation point A is 18,66 km.

Feedback to Self-Check Exercises

Lesson 1

1. a) $x^2 - 11x + 28 = 0$

$\therefore (x-7)(x-4) = 0$

$\therefore x = 7 \quad or \quad x = 4$

b) $4x^2 - 8x - 5 = 0$

$\therefore (2x-5)(2x+1) = 0$

$\therefore x = \frac{5}{2} \quad or \quad x = -\frac{1}{2}$

c) $x^2 - 14x + 45 = 0$

$\therefore (x-9)(x-5) = 0$

$\therefore x = 9 \quad or \quad x = 5$

d) $3x^2 + 5x + 1 = 0$

$$\therefore x = \frac{-(5) \pm \sqrt{(5)^2 - 4(3)(1)}}{2(3)}$$

$$\therefore x = \frac{-5 \pm \sqrt{13}}{6}$$

Questions 1a to 1c could be solved using the quadratic equation formula if you prefer. Question 1d, however, **must** be solved using the formula since the trinomial does not factorise and so solving the equations using factors is not an option.

e) $3x^2 - 3x + 1 = 0$

$$\therefore x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(1)}}{2(3)}$$

$$\therefore x = \frac{3 \pm \sqrt{-3}}{6}$$

$\therefore$ there is no solution

2. a) $x = \frac{3}{2} \quad or \quad x = \frac{1}{2}$

$\therefore 2x - 3 = 0 \quad or \quad 2x - 1 = 0$

$\therefore (2x-3)(2x-1) = 0$

$\therefore 4x^2 - 8x + 3 = 0$

2. b) $x=\frac{3}{2}\quad or \quad x=-\frac{1}{2}$

$\therefore 2x-3=0 \quad or \quad 2x+1=0$

$\therefore (2x-3)(2x+1)=0$

$\therefore 4x^2-4x-3=0$

c) $x=-\frac{3}{2}\quad or \quad x=\frac{1}{2}$

$\therefore 2x+3=0 \quad or \quad 2x-1=0$

$\therefore (2x+3)(2x-1)=0$

$\therefore 4x^2+4x-3=0$

d) $x=-\frac{3}{2}\quad or \quad x=-\frac{1}{2}$

$\therefore 2x+3=0 \quad or \quad 2x+1=0$

$\therefore (2x+3)(2x+1)=0$

$\therefore 4x^2+8x+3=0$

e) $x=3 \quad or \quad x=3$

$\therefore x-3=0 \quad or \quad x-3=0$

$\therefore (x-3)(x-3)=0$

$\therefore x^2-6x+9=0$

3. a) $2x+\frac{1}{x}=3$

$\therefore 2x^2+1=3x$

$\therefore 2x^2-3x+1=0$

$\therefore (2x-1)(x-1)=0$

$\therefore x=\frac{1}{2}\quad or \quad x=1$

b) $\frac{x}{x-1}+\frac{2}{x-2}=3$

$\therefore x(x-2)+2(x-1)=3(x-1)(x-2)$

$\therefore x^2-2x+2x-2=3(x^2-3x+2)$

$\therefore x^2-2=3x^2-9x+6$

$\therefore -2x^2+9x-8=0$

$\therefore 2x^2-9x+8=0$

*We are forced to use the **Quadratic Equation Formula** here since the trinomial does not factorise.*

$\therefore x=\frac{-(-9)\pm\sqrt{(-9)^2-4(2)(8)}}{2(2)} \qquad \therefore x=\frac{9\pm\sqrt{17}}{4}$

4. Let the size of plot A be x ha.

$\therefore$ the size of plot B is $(x+45)$ ha.

For plot A, productivity per hectare = $\frac{680}{x}$

For plot B, productivity per ha = $\frac{680}{x+45}$

$$\therefore \frac{680}{x} - \frac{680}{x+45} = 9$$
$$\therefore 680(x+45) - 680x = 9x(x+45)$$
$$\therefore 9x^2 + 405x - 30600 = 0$$
$$\therefore x^2 + 45x - 3400 = 0$$
$$\therefore (x+85)(x-40) = 0$$
$$\therefore x = -85 \quad or \quad x = 40$$

You may not find the factors of 3 400 easily. Try a number of options. If you can't find the factors you can always use the formula to find the solutions

Since area cannot be negative, $x = 40$ is the answer.

Plot A is 40 ha and plot B is 85 ha.

The question is about productivity, so:

Productivity for plot A = $\frac{680}{40} = 17$ tons per ha

Productivity for plot B = $\frac{680}{85} = 8$ tons per ha

Lesson 2

1. $LHS = (\cos x + \sin x)^2 + (\cos x - \sin x)^2$

$$= \cos^2 x + 2\cos x \sin x + \sin^2 x + \cos^2 x - 2\cos x \sin x + \sin^2 x$$
$$= 2\cos^2 x + 2\sin^2 x$$
$$= 2(\cos^2 x + \sin^2 x)$$
$$= 2$$
$$= RHS$$

Pythagorean Identity:

$$1 - \sin^2 x = \cos^2 x$$

2. $LHS = \cos(A+B)\cos(A-B)$

$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$

$= \cos^2 A - \cos^2 A\sin^2 B - \sin^2 B + \cos^2 A\sin^2 B$

$= \cos^2 A - \sin^2 B$

RHS

3. $LHS = \dfrac{\sin A\sin 2A}{\cos A} + \cos 2A$

$= \dfrac{\sin A.2\sin A\cos A}{\cos A} + \cos^2 A - \sin^2 A$

$= \dfrac{2\sin^2 A\cos A}{\cos A} + \cos^2 A - \sin^2 A$

$= 2\sin^2 A + \cos^2 A - \sin^2 A$

$= \sin^2 A + \cos^2 A$

$= 1$

$= RHS$

Lesson 3

1. $2\sin^2 x = \cos 2x$

$\therefore 2\sin^2 x = \cos^2 x - \sin^2 x$

$\therefore 3\sin^2 x = \cos^2 x$

$\therefore \dfrac{3\sin^2 x}{\cos^2 x} = \dfrac{\cos^2 x}{\cos^2 x}$

$\therefore 3\tan^2 x = 1$

$\therefore \tan^2 x = \dfrac{1}{3}$

$\therefore \tan x = \pm\dfrac{1}{\sqrt{3}}$

$\therefore x = 30^\circ + k180^\circ$ or $x = -30^\circ + k.180^\circ$ where $k \in Z$

2. $5\sin x - 2 = 1 - \cos 2x$

$$5\sin x - 2 = 1 - \left(\cos^2 x - \sin^2 x\right)$$

$$\cos^2 x - \sin^2 x + 5\sin x - 3 = 0$$

$$\left(1 - \sin^2 x\right) - \sin^2 x + 5\sin x - 3 = 0$$

$$-2\sin^2 x + 5\sin x - 2 = 0$$

$$2\sin^2 x - 5\sin x + 2 = 0$$

$$\left(2\sin x - 1\right)\left(\sin x - 2\right) = 0$$

$\therefore \sin x = \frac{1}{2}$ or $\sin x = 2$ invalid

$\therefore x = 30^\circ + k\ 360^\circ$ or $x = 150^\circ + k360^\circ$ where $k \in Z$

Lesson 4

First draw two right-angled triangles with (i) 45° and (ii) 30° and 60° to help in answering the questions.

1. a) $\left(\frac{\sqrt{3}}{2}\right)^2 + (1)^2 = \frac{3}{4} + 1$

$$= \frac{7}{4}$$

b) $1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4}$

$$= \frac{3}{4}$$

2. a) $\cos^2\theta = \frac{1}{2}$

$\therefore \cos\theta = \pm\frac{1}{\sqrt{2}}$

$\therefore \theta = \pm 45^\circ + k.360^\circ$ or $\theta = \pm 135^\circ + k.360^\circ$ where $k \in Z$

$\therefore \theta = 45^\circ, -45^\circ, 315^\circ, 135^\circ, -135^\circ, 225^\circ$

b) $\therefore \dfrac{\sin\theta}{\cos\theta}.2\sin\theta\cos\theta = 0$

$\therefore 2\sin^2\theta = 0$

$\therefore \sin\theta = 0$

$\therefore \theta = 0^\circ + k.360^\circ$ or $\theta = 180^\circ + k.360^\circ$ where $k \in \gtrless$

$\therefore \theta = 0^\circ, 360^\circ, 180^\circ, -180^\circ.$

3. a) $\sqrt{1+\cos\theta} = 2\sin\theta$

$\therefore 1+\cos\theta = 4\sin^2\theta$

$\therefore 1+\cos\theta = 4\left(1-\cos^2\theta\right)$

$\therefore 4\cos^2\theta + \cos\theta - 3 = 0$

$\therefore (4\cos\theta - 3)(\cos\theta + 1) = 0$

$\therefore \cos\theta = \dfrac{3}{4}$ or $\cos\theta = -1$

$\therefore \theta = \pm 41{,}41^\circ + k.360^\circ$ or $\theta = \pm 180^\circ + k.360^\circ$ where $k \in Z$

$\therefore \theta = 41{,}41^\circ, -318.59^\circ, -41{,}41^\circ, 318{,}59^\circ, 180^\circ, -180^\circ$

b) $2\sin\theta - \cos\theta = 0$

$\therefore 2\sin\theta = \cos\theta$ (dividing both sides of the equation by $2\cos\theta$)

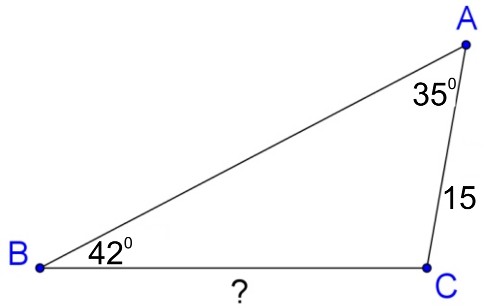
$\therefore \tan\theta = \dfrac{1}{2}$

$\therefore \theta = 26{,}57^\circ + k.180^\circ$ where $k \in Z$

$\therefore \theta = 26{,}57^\circ, 206{,}57^\circ, -153{,}43^\circ, -333{,}43^\circ$

Lesson 5

1.

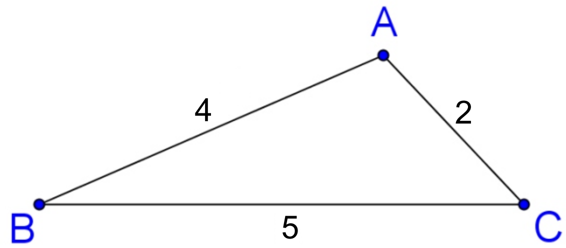


$$\frac{a}{\sin 35^{\circ}} = \frac{15}{\sin 42^{\circ}}$$

$$\therefore a = \frac{15\sin 35^{\circ}}{\sin 42^{\circ}}$$

$$= 12{,}86\ m$$

2.



$$5^2 = 4^2 + 2^2 - 2(4)(2)\cos A$$

$$\therefore \cos A = \frac{20-25}{16}$$

$$\therefore \hat{A} = 108{,}21^{\circ}$$

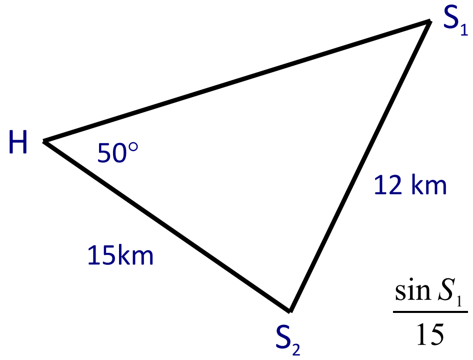
$$4^2 = 5^2 + 2^2 - 2(5)(2)\cos C$$

$$\therefore \cos C = \frac{29-16}{20}$$

$$\therefore \hat{C} = 49{,}46^{\circ}$$

$$\therefore \hat{B} = 22{,}33^{\circ}$$

3.



$$\frac{\sin S_1}{15} = \frac{\sin 50^\circ}{12}$$

$$\therefore \sin S_1 = \frac{15\sin 50^\circ}{12}$$

$$\therefore \hat{S}_1 = 73{,}25^\circ \quad or \quad \hat{S}_1 = 106{,}75^\circ$$

$\therefore S_2 = 56{,}75^\circ$ or $S_2 = 23{,}25^\circ$

$\therefore \dfrac{HS_1}{\sin 56{,}75^\circ} = \dfrac{12}{\sin 50^\circ}$ or $\dfrac{HS_1}{\sin 23{,}25^\circ} = \dfrac{12}{\sin 50^\circ}$

$\therefore HS_1 = \dfrac{12\sin 56{,}75^\circ}{\sin 50^\circ}$ or $HS_1 = \dfrac{12\sin 23{,}25^\circ}{\sin 50^\circ}$

$= 13{,}10\,km$ $\qquad = 6{,}18\,km$

Therefore, the slower ship is travelling at about 13km/h or 6 km/h.