Mathematics

Unit 4

The Calculus

Acknowledgements:

Editor: Allison Kitto
Writer: Allison Kitto, Rob Houghton-Rhodes
Layout: Lidia Kruger

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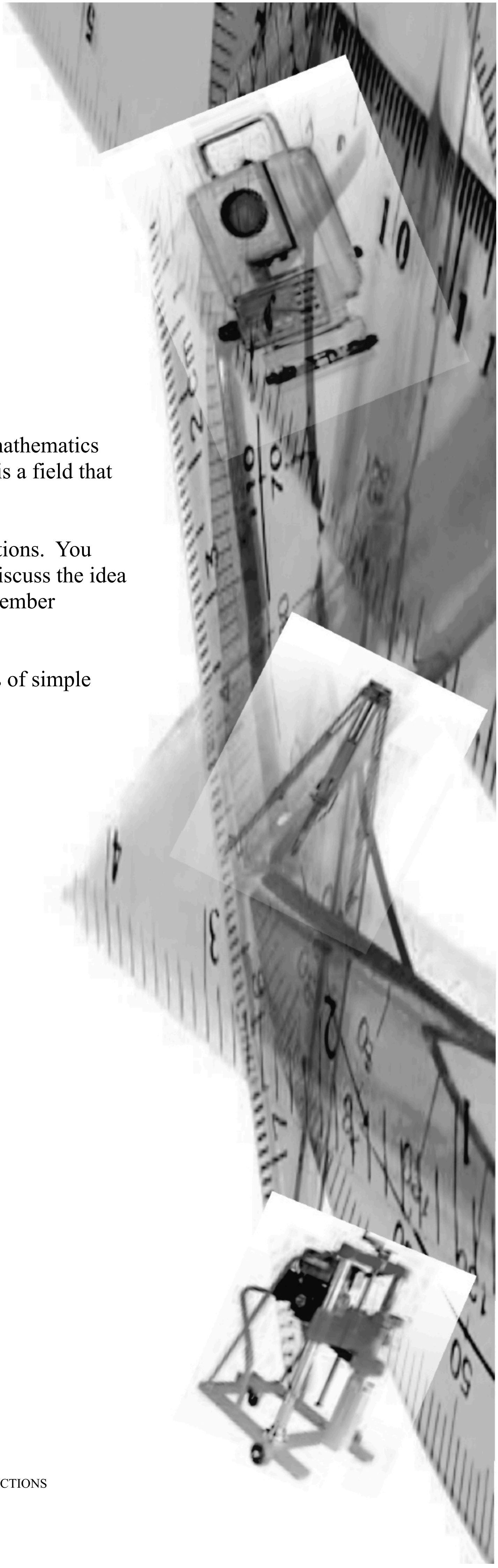
LESSON 1

Limits of Functions



About this lesson

In this unit you are going to be introduced to a field of mathematics that may be new to you. The field is called calculus. It is a field that is widely used in mathematics and other sciences.

In this lesson you are going to learn about limits of functions. You already know what functions are. We are first going to discuss the idea of a limit using functions. In this lesson you need to remember functions and the slope or gradient of a function.

At the end of the lesson you should be able to find limits of simple functions.

In this lesson you will:

- find the values of functions
- define limits
- calculate limits of various functions
- evaluate ratios of small quantities
- simplify limits

Values of functions

Let us start with something that you know already.

Functions are usually written as $f(x) = ...$. Sometimes we forget that $f(x)$ is actually a value. In our coordinate system we have x and y –coordinates, usually written as $(x; y)$. In this form each y –coordinate actually stands for $f(x)$. For example, $f(x) = 3x + 4$ can also be written as $y = 3x + 4$. It is common therefore to write $y = f(x)$. It is important to realise that this $f(x)$ is the value of the function at x.

Let us look again at the function $f(x) = 3x + 4$. When you make a table of points to plot a graph, it looks as follows:

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	–11	–8	–5	–2	1	4	7	10	13	16	19

Since $y = f(x)$, everyone understands that all the y –coordinates stand for the values of the function at the various points with the given x –coordinates. It is also important to remember that for convenience, we usually only choose integer x –values for the table. However, the function exists at all points between those integers. That is why we draw a continuous line for this graph.

Sometimes it is not possible to get a value of a function at a given point. The function may just not exist at that point. An example of such a function is

$$f(x) = \frac{1}{x}$$

This function has values for all x except where $x = 0$. Can you see why? Because, at $x = 0$, the function is not defined. Division by zero is not admissible. We say the curve is not continuous, or is discontinuous, at $x = 0$. Can you think of other examples where such a thing happens? Some examples are:

a) $f(x) = \dfrac{3+x}{x}$ at $x = 0$

b) $f(x) = \dfrac{1}{x+3}$ at $x = -3$

c) $f(x) = \dfrac{x}{x-1}$ at $x = 1$

We may think of the curve as being 'broken' at points like these. You may have noted that the examples above are all those were the denominator is equal to zero for a certain value of x. There are many such examples, but we will only do a few easy ones.

To help you analyse what is happening here, do the following activity.

ACTIVITY 1

Make tables of coordinates for the following functions and draw the graphs.

a) $f(x) = \dfrac{1}{x}$ b) $f(x) = \dfrac{x}{x - 1}$

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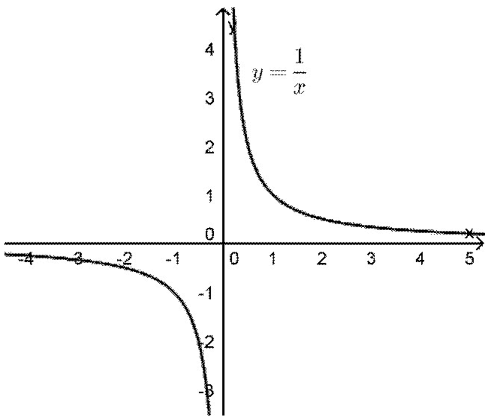
In both these drawings there is a vertical line which the graphs approach that goes through the x –value where the graph does not exist. This line is called **an asymptote**. The graph does not go through the points with such x –coordinates. In this way we illustrate that the function is not continuous at that point. Note that we draw a dotted line for an asymptote.

In each drawing as the x –values approach the value where there is an asymptote, the curve approaches the asymptote in different ways. This is what we are going to discuss in the next two sections.

Concept of a limit

In the previous activity each function has a value at which it is not defined (does not exist). However, it is possible to calculate the values of the functions, $f(x)$, as x gets closer and closer and closer to the values at which the functions are nor defined. Notice, from the graphs, that in each case the function values behave completely differently as those values are approached from the left-hand side compared with when they are approached from the right-hand side.



ACTIVITY 2

Consider the function $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

1. For which value of x is $f(x)$ not defined?

2. Is it possible to calculate $f(2)$?

3. Complete the following table:

x	1,9	1,99	1,999	1,999	2	2,0001	2,001	2,01	2,1
$f(x)$					?				

4. Complete:
   a) As x approaches 2 from the left side $f(x)$ appears to approach ____________________

   b) As x approaches 2 from the right side $f(x)$ appears to approach _________________________

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5. Sketch the graph of $y = f(x)$

In Activity 1, $f(x)$ appears to be approaching the number 3 as x approaches 2. We call the number 3 'the limit of $f(x)$ as x approaches 2' and write:

$f(x) \to 3$ as $x \to 2$
or
$\lim_{x \to 2} f(x) = 3$

ACTIVITY 3

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Suppose you are given the function $g(x) = 2x$. Find the limit of the function $g(x)$ as the variable x approaches 2, i.e. $\lim_{x \to 2} g(x)$.

ACTIVITY 4

Write the following expressions using limit notation, and find the limits of the functions as x approaches the value given in brackets.

1. $f(x) = 5x$ (3)

2. $f(x) = 3x - 1$ (5)

3. $y = 5x + 2$ (5,2)

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4. $y = 3x^2 + 4x - 2$ (3)

There are some very complicated limits. What happens when we have functions whose graphics have asymptotes? In order to understand this question, we will look at ratios of small numbers.

Ratios of small quantities

Let us look back at the function $f(x) = \frac{1}{x}$

What happens as x approaches zero? Answer this question by doing the following activity.

ACTIVITY 5

Fill in the following tables:

x	–0,1	–0,05	–0,01	–0,001	–0,0005	–0,0001
$f(x)$						

x	0,1	0,05	0,01	0,001	0,0005	0,0001
$f(x)$						

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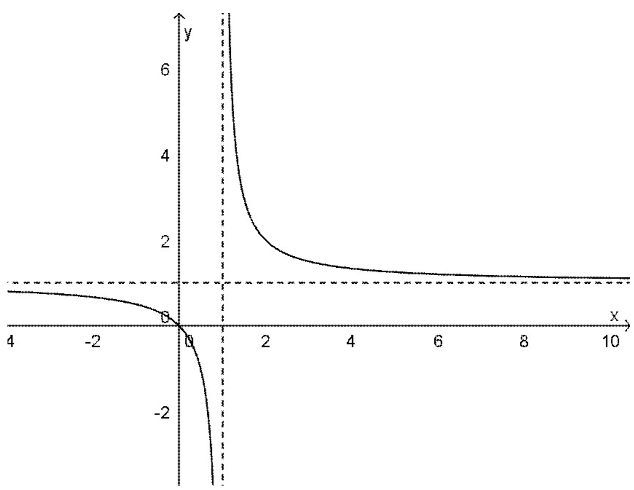
Are these values approaching the same number? What does this mean?

The values of the function do not approach the same number as x approaches zero. As x approaches 0 from the left, they get 'bigger', but negative. As x approaches 0 from the right, they also get bigger, but remain positive. Let us see how this can be argued.

The numerator of this function is a constant positive number. This means that the sign of the answer depends entirely on the sign of the denominator. If the sign of the denominator is negative, the answer is negative. If the denominator is positive, the answer is positive.

Graphically, this can be drawn as follows:



The figure on the previous page shows the graph of the function $f(x) = \frac{1}{x}$.

The sketch shows what happens to the values of the function as x approaches 0 from both sides, that is from the left, and from the right.

Some notation:

1. We use the symbol ∞ ('infinity') to indicate that positive numbers are increasing without stopping, and –∞ ('negative infinity') to indicate that negative numbers are decreasing without stopping.

2. We write $x \to a^+$ to indicate that x approaches the number a 'from the right' i.e. through numbers greater than a, and $x \to a^-$ to indicate that x approaches the number a 'from the left' i.e. through numbers less than a.

For our function $f(x) = \frac{1}{x}$ we can write

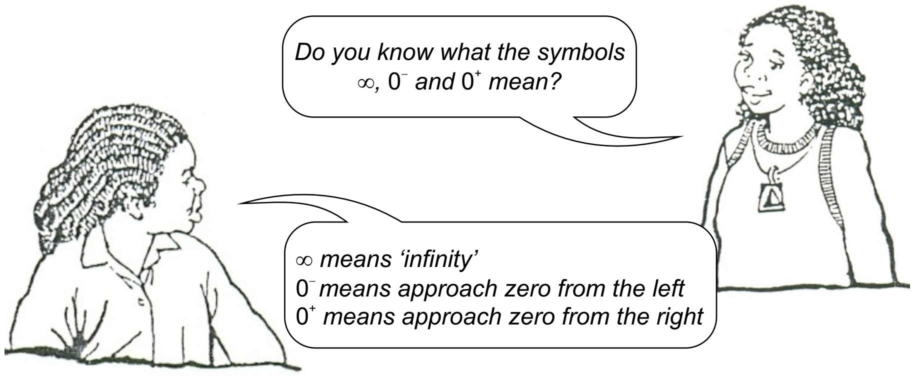
$$f(x) \to \infty \quad \text{as } x \to 0^+$$

and $f(x) \to -\infty$ as $x \to \bar{0}$

Can you see this from the graph in the figure on the previous page?

We can also write $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to 0^-} f(x) = -\infty$

It is very important to understand that ∞ is **not** a real number. Think of ∞ as meaning 'gets bigger and bigger and ...'.



ACTIVITY 6

Sketch the graph of the function for values of to the left and right of $x = 3$.

$$f(x) = \frac{x}{x-3}$$

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The emphasis in the drawing of this graph is only to show the most important points of the graph.

The graph shows where the x –intercept of the function is. It also show where the y –intercept is.

Most importantly, it shows where the function does not exist. Further, it shows what happens to the values of the function as x approaches 3 from the right and from the left.

Again the argument is as follows:

For all positive x the numerator is positive. For all x less than 3, i.e. left of 3, the denominator, $x - 3$, is negative. As we go closer to 3, the difference, $x - 3$, becomes very small, but negative. The expression

$$\frac{x}{x-3} \to -\infty$$

i.e. becomes a very big negative number. This can be written as follows:

$$\lim_{x \to 3^-} \frac{x}{x-3} = -\infty$$

For all values greater than 3 the denominator, $x-3$, is positive. As we go closer to 3, the difference, $x-3$, becomes very small, but positive. The expression

$$\frac{x}{x-3} \to \infty$$

i.e. becomes a very big positive number. This can be written as follows:

$$\lim_{x \to 3^+} \frac{x}{x-3} = +\infty$$

Notice when we use $\to$ and when we use =. Do not confuse the notation.

Simplifying limits

Sometimes it is necessary to check whether there is anything that can be done to the given expression before you start calculating the limit. We can explain this by means of an example:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

If you substitute for $x = 3$ directly in this expression, you will get

$$\frac{0}{0}$$

What is this? Obviously you cannot say what the answer is. Do not forget, division by zero is not admissible. So, what can you do? Try the following activity to find out.

ACTIVITY 7

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Make a table of the values of this function as *x* approaches 3 from the left and from the right. What do you note from the results? Make a rough sketch of what you found.

From the results we see that the values of the function are approaching the value 6, from both the left and rom the right. One could claim therefore, that

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$$

How do you get this answer more quickly without going through the steps of using calculators and writing out a table? The first thing is to note that we can factorise the numerator of the expression.

$$x^2 - 9 = (x + 3)(x - 3)$$

We can therefore write the function as follows:

$$f(x) = \frac{(x+3)(x-3)}{x-3}$$

Do you see what the next step can be? Of course, this form of writing the function makes it possible to simplify the limit this way:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} \qquad \text{[cancel } (x-3)\text{]}$$

$$\lim_{x \to 3} (x + 3)$$

$$= 6$$

Remember, you should first check whether there is any way in which you can simplify the given expression.

Try the following activity.

ACTIVITY 8

1. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

2. $\lim_{x \to -3} \frac{x^2 + 7x + 12}{x + 3}$

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Summary

Recall what you have learnt in this lesson:

- You were first reminded that $f(x)$ stands for values of functions at given values of x, the independent variable. You found out that some functions do not exist at certain points, and these functions are said to be discontinuous at those points.

- When a function is continuous at a point you can simply substitute to find the limit.

- You then learnt how to find limits of various functions. You learnt that the limit of a function as x approaches a number a can be represented as follows:

  $\lim_{x \to a} f(x) = l$ or $f(x) \to l$ as $x \to a$

- You found that some limits require an understanding of ratios of small quantities. You found that you can use this understanding of ratios of small quantities to find limits of quotients such as

$$\lim_{x \to a} \frac{1}{x}$$

- You found that it is important to understand that the limit of a function as x approaches a certain value may be different as x approaches the number from the left and from the right. So, you now understand the limit of a function $f(x)$ as x approaches a number a from the left

$$\lim_{x \to a^-} f(x)$$

  and the limit of a function $f(x)$ as approaches a number a from the right

$$\lim_{x \to a^+} f(x)$$

- Lastly you found that sometimes there is a need to simplify the given expression before calculating a limit. Although there are other methods that can be used to simplify functions factorisation is the most commonly used and is the easiest.

CHECKLIST

Are you able to:

- use an asymptote to indicate an appropriate point of discontinuity
- explain $\lim_{x \to 0} \frac{1}{x}$ graphically
- evaluate limits by factorisation and cancelling

Now try the following exercise to check yourself.

SELF-CHECK EXERCISE

1. Find the limits of the following functions as x approaches the number in brackets:

   a) $f(x) = 3x + 2$ (0)

   b) $f(x) = x^2 - 2x + 6$ (2)

   c) $f(x) = 3x^2 - 6x - 8$ (4)

   d) $f(x) = 5$ (0)

2. Use your calculator to find the limits of the function as x approaches 2 from the left and from the right:

   $f(x) = \dfrac{x+2}{x-2}$

3. Find the limit of the function as x approaches the number in brackets:

   a) $f(x) = \dfrac{x^2 + x - 6}{x - 2}$ (2)

   b) $f(x) = \dfrac{3x^2 + 4x - 4}{x + 2}$ (–2)

4. If $f(x) = x^2 + x - 1$, find

   a) $f(2)$

   b) $f(\phi)$

   c) $f(1+h)$

   d) $f(x+h)$

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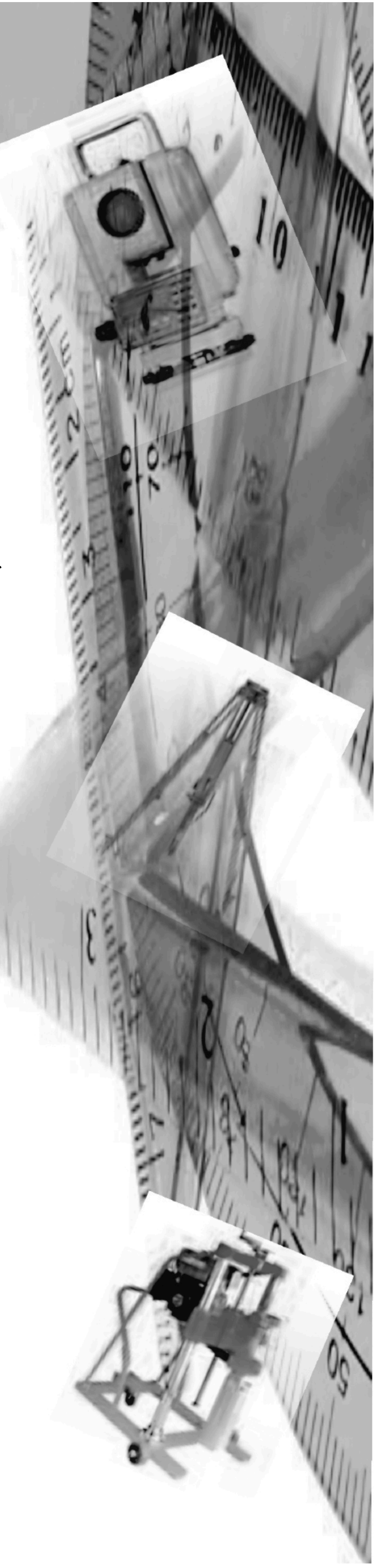
LESSON 2

Introduction to the Calculus



About this lesson

In the previous lesson we discussed the idea of functions that increase or decrease, gradually approaching a particular point or value. This lesson will take that idea further. We will be discussing a new field called the calculus. We will start with a discussion on the problems that the calculus deals with. Then we will introduce the calculus through the idea of a gradient. This is extended to include gradients of curves. To do this we will use limits.

For this lesson you will need to remember the formula for finding gradients of straight lines. You will also need to know something about ratios. The previous lesson dealt with ratios of small numbers. That will also be useful.

In this lesson you will:

- find the average gradient between two points on a curve
- find the gradient of a curve at a point on the curve using limits
- differentiate functions of the type $b, x, x^2, x^{\wedge}, \frac{1}{x}, x^n$; and combinations of these functions
- discover some problems which are solved using the calculus
- investigate average gradients
- calculate the gradient of a curve at a point
- find the derivative of a function rom first principles and by inspection
- learn rules for finding derivatives

Problems solved using the calculus

The problem of getting the most, or the best, out of any deal is very old. Here is a story told by one of the greatest poets of ancient Rome, Publius Maro:

The Phoenician princess, Dido, fled after being persecuted by her brother. She went westward along the Mediterranean Sea.

When she arrived at a spot, now called the bay of Tunis, she bought land from Yarb, the local leader. She asked for land that could be covered by a bull's hide. Dido then cut a bull's hide in to narrow strips and tied them together. This became a very long strip that enclosed a very large piece of land. On this land she built a fortress and lived happily. In mathematics this is called optimisation.

In geometry, we talk of maximum area and minimum perimeter problems. In industry there are usually problems like this:

> Find the minimum surface area of metal that can make a metallic cylinder (a can) which has a volume of 1 litre.

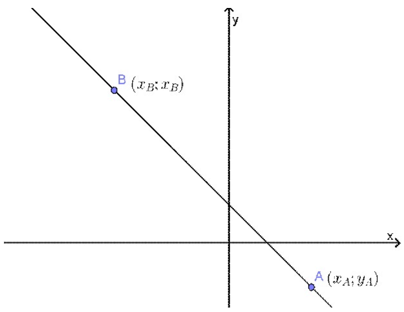
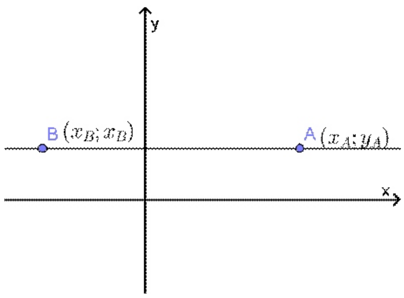
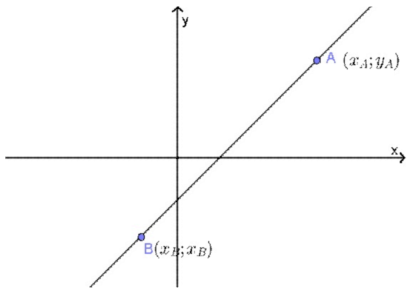
The aim of such problems is to minimise the amount of material used to make cans. Calculus can be used to solve such problems.

Did you know that light always **chooses** a path that takes the least time to cover any distance? This minimisation of time explains issues like refraction of light. Calculus can therefore also be used in optics.

Average rate of change

Do you remember that the gradient (or slope) of a line passing through the points (x_A; y_A) and (x_B; y_B) is defined by

$$
\begin{aligned}
m &= \frac{y_B - y_A}{x_B - x_A} \\
&= \frac{\text{change in } y}{\text{corresponding change in } x} \\
&= \frac{\Delta y}{\Delta x} \quad \text{(read: delta } y \text{ over delta } x\text{)}
\end{aligned}
$$

(a)	(b)	(c)
		
$m = \frac{y_B - y_A}{x_B - x_A} \; \frac{(<0)}{(>0)}$	$m = \frac{y_B - y_A}{x_B - x_A} \; \frac{(=0)}{(>0)}$	$m = \frac{y_B - y_A}{x_B - x_A} \; \frac{(>0)}{(>0)}$
< 0 (negative) The line slopes downward from left to right i.e. y decreases as x increases The function is 'decreasing'.	= 0 The line is horizontal i.e. y remains constant as x increases.	> 0 (positive) The line slopes upward from left to right i.e. y increases as x decreases The function is 'increasing'.

Note:

1. When we describe a function as 'increasing' or 'decreasing' we always mean from left to right i.e. as x increases.
2. The gradient of a line is constant, i.e. has the same value between any two points on the line. Our aim now is to develop the concept of a gradient for any curve.

ACTIVITY 1

a) Sketch the parabola $y = x^2 - 4$.

b) Calculate the gradient of, and draw the line segments joining the following points on the parabola:

(i) (–2; 0) and (–1; –3)
(ii) (–2; 0) and (0; –4)
(iii) (–2; 0) and (1; –3)

(iv) (–2; 0) and (2; 0)
(v) (–2; 0) and (3; 5)

c) 'On average', has the parabola decreased, remained constant or increased between each of the points given in (i) to (v) in b)?

d) Does the parabola actually decrease, remain constant or increase between the points given in (i) to (v) in b)? Discuss your answers.

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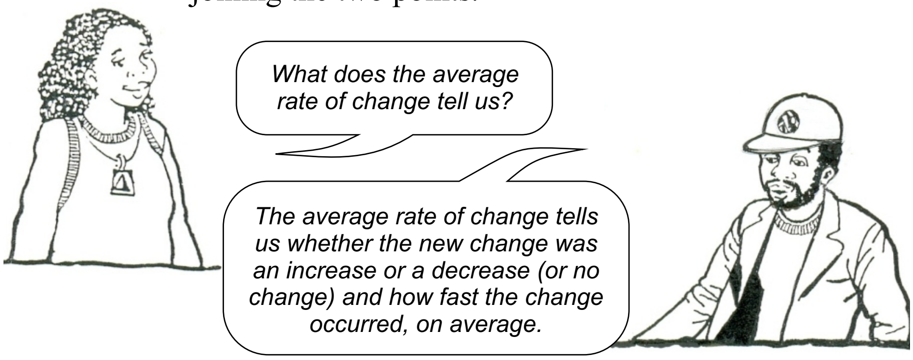
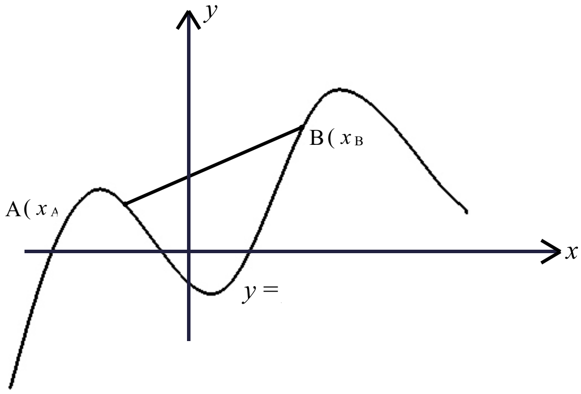
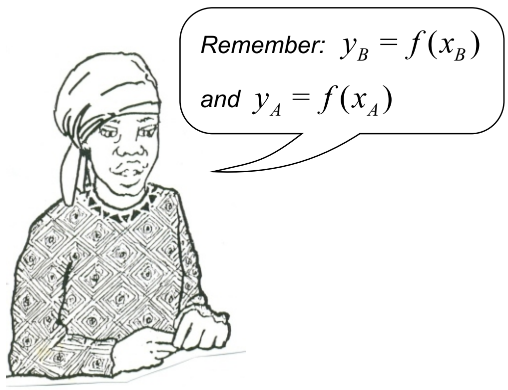
Definition: The average rate of change of a function between two points on its graph is the gradient of the line segment joining the two points.



Notation:





The average rate of change, or average gradient, of the function f between $A(x_A; y_A)$ and $B(x_B; y_B)$ (as shown) is

$$\frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A} \quad (x_A \neq x_B)$$

$$= \frac{f(x_B) - f(x_A)}{x_B - x_A}$$

If $x_B - x_A = h$, , then $x_B = x_A + h$

then the average gradient of f between $A(x_A; y_A)$ and $B(x_B; y_B)$ is

$$\frac{f(x_A + h) - f(x_A)}{h}$$

ACTIVITY 2

Use the formula: average gradient $= \dfrac{f(x_1+h)-f(x_1)}{h}$

to find the average gradient of the function $f(x) = 2x^2 - x + 1$
between the points with x –coordinates.

a) $x_1 = 0$ and $x_2 = 2$

b) $x_1 = 0$ and $x_2 = 1$

c) $x_1 = 0$ and $x_2 = \dfrac{1}{2}$

d) $x_1 = 0$ and $x_2 = \dfrac{1}{4}$

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Summary

- The average gradient, or average rate of change, of a function defined by $y = f(x)$ between two points $A(x_A; y_A)$ and $B(x_B; y_B)$ on the graph of the function is given by

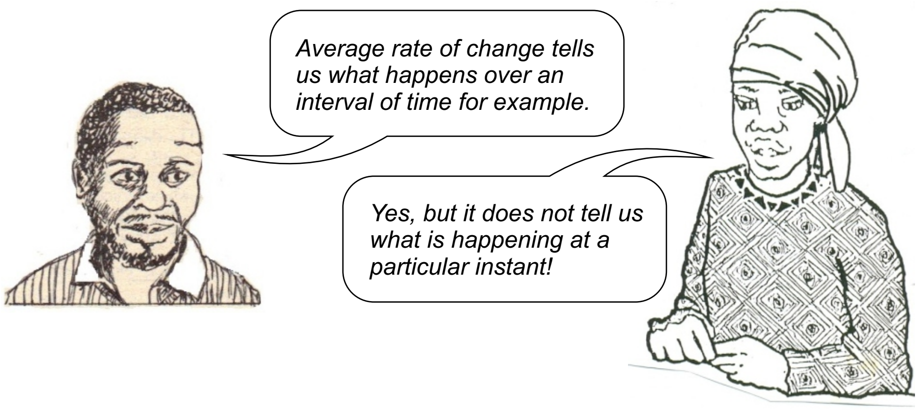
$$\frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B = x_A} \qquad (x_A \neq x_B)$$

$$= \frac{f(x_B) - f(x_A)}{x_B - x_A}$$

$$= \frac{f(x_A + h) - f(x_A)}{h} \qquad (h = x_B - x_A)$$

- Average gradient or average rate of change, tells us the average or net change of a function over an interval and is useful, for example, in the study of climate change, population growth and economics.

- The average gradient or average rate of change between two values can be calculated either by substituting the values into the function and then calculating the average gradient (Method 1) or by deriving a formula for the average gradient using the function and then substituting the values into the formula (Method 2).

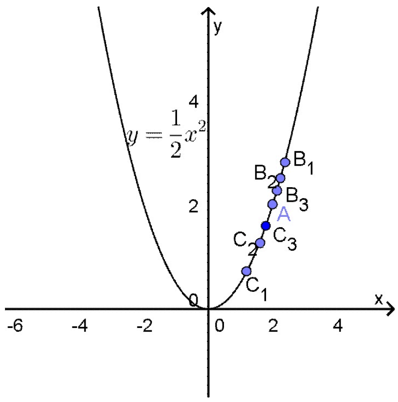


The gradient of a curve at a point

Instead of the average change of a function between two points (i.e. over an interval) we often want to know how the function is changing or behaving at a single point (i.e. at that particular instant; this is known as the instantaneous rate of change at the point). To do this we use what we learnt about limits of functions in Lesson 1. You may wish to revise the ideas and notation we learnt before continuing with this section.

Example 1

Consider the function defined by $f(x)=\frac{1}{2}x^2$. We are going to try and find the 'gradient' of this function at the point where $x = 2$ (i.e. at the point A (2; 2) on the graph $y=\frac{1}{2}x^2$).



Our strategy is to sneak up on A by choosing points close to A and calculating the average gradients between those points and A.

Since we want to calculate several average gradients we shall use 'Method 2' and first calculate a formula in terms of h for the average gradient between A (2; 2) and any other point $B(x_B\,;\,y_B)$ on the graph $y=\frac{1}{2}x^2$.

Let $h = x_B - x_A = x_B - 2$ then $x_B = 2 + h$

and $y_B = f(x_B) = f(2+h) = \frac{1}{2}(2+h)^2 = 2 + 2h + \frac{1}{2}h^2$

Average gradient of f between A (2; 2) and $B(x_B; y_B)$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A}$$

$$= \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2 + 2h + \frac{1}{2}h^2 - (2)}{h}$$

$$= 2 + \frac{1}{2}h$$

We now draw up a table using points $B_1, B_2, B_3, \ldots$ on $y = \frac{1}{2}x^2$ each getting closer and closer to A.

$B(x_B; y_B)$	$h = x_B - x_A$	$\frac{y}{x} = 2 + \frac{1}{2}h$
$B_1(3; f(3))$	$h = 3 - 2 = 1$	$\frac{y}{x} = 2 + \frac{1}{2}(1) = 2,5$
$B_2(2,5; f(2,5))$	$h = 2,5 - 2 = 0,5$	$\frac{y}{x} = 2 + \frac{1}{2}(0,5) = 2,25$
$B_3(2,1; f(2,1))$	$h = 2,1 - 2 = 0,1$	$\frac{y}{x} = 2 + \frac{1}{2}(0,1) = 2,05$
$B_4(2,01; f(2,01))$	$h = 2,01 - 2 = 0,01$	$\frac{y}{x} = 2 + \frac{1}{2}(0,01) = 2,005$
$B_5(2,001; f(2,001))$	$h = 2,001 - 2 = 0,001$	$\frac{y}{x} = 2 + \frac{1}{2}(0,001) = 2,0005$
$B_6(2,0001; f(2,0001))$	$h = 2,0001 - 2 = 0,0001$	$\frac{y}{x} = 2 + \frac{1}{2}(0,0001) = 2,00005$

It would seem that as the points $B_1, B_2, B_3, \ldots$ move closer and closer to A the average gradient of the line (secant) joining them, gets closer and closer to 2.

ACTIVITY 3

In the example we considered only points to the right of A (2; 2). Now do the same for points $C_1(1; f(1))$, $C_2(1,5; f(1,5))$, $C_3(1,9; f(1,9))$, $C_4(1,99; f(1,99))$, $C_5(1,999; f(1,999))$, $C_6(1,9999; f(1,9999))$, ... which are all to the left of A (2; 2).

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Once again as we approach A (2; 2), but this time from the left, the average gradients seem to approach the number 2.

From our study of limits in Lesson 1, none of this should be surprising. All we are doing is allowing h to approach (tend to) 0 in the expression $\frac{f(2+h) - f(2)}{h}$

i.e we are calculating $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

So, let's do it:

$$\lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{1}{2}(2+h)^2 - \frac{1}{2}(2)^2}{h}$$
$$= \lim_{h \to 0} \frac{[\frac{1}{2}(4+4h+h^2)]-[2]}{h}$$
$$= \lim_{h \to 0} (2+\frac{1}{2}h)$$
$$= 2$$

As we approach A (2; 2) along the graph $y = \frac{1}{2}x^2$, from both the left and the right, the average gradients approach the number 2.

We say that the gradient of $y = \frac{1}{2}x^2$ at A (2;2) is 2.

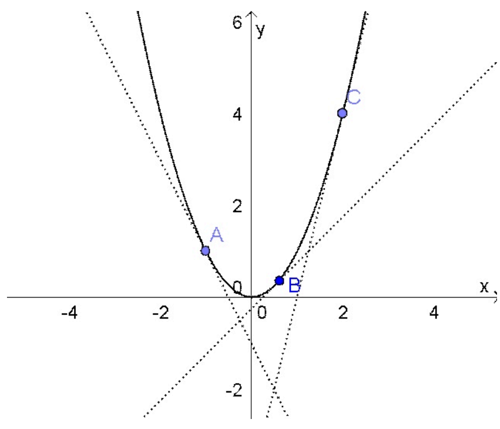
We also say that the instantaneous rate of change of $y = \frac{1}{2}x^2$ at A (2; 2) is 2.

The gradient of a curve at a point

Geometrically, the gradient of any curve at any point can be regarded as the gradient of the tangent at that point. Do you know what a tangent to a curve at a point on the curve is? A tangent to a curve at a point on the curve is a straight line that touches the curve at that point. It shows the slope or gradient of the curve at that point.

Look at the adjacent diagram.

This diagram shows a curve with points A, B and C chosen along the curve. The tangent at a point A has a different gradient from the tangent at point B, which is also different from the gradient of the tangent at point C. It is therefore not possible to talk of the gradient of a curve as a constant gradient of a curve is not constant, it changes. We need to find a way of expressing the gradient of a curve at each of its points.



ACTIVITY 4

Find the gradient of the function $f(x) = x^2$ at the point $x = 1$.

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Here is another activity.

ACTIVITY 5

What is the slope of $f(x) = 3x^2 + 4$ at $x = 3$?

ANSWERS ON PAGE 53

Is it possible to find the gradient of a curve at any point on the curve? Yes. We only have to use , instead of the numbers. Try the following activity.

ACTIVITY 6

Find the gradient of the function $f(x) = x^2$ at any point on its graph.

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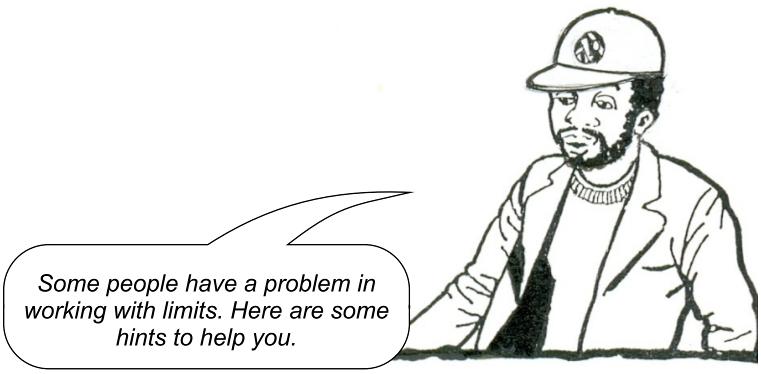

- Do not find limits until you have simplified the arithmetic. In the case of a gradient, you have to simplify the ratio $\dfrac{f(x+h)-f(x)}{h}$
- The limit of the simplified form is easy to find. You only have to let $h \to 0$ which, in most cases, will be the same as substituting $h = 0$.

Now try the following activity.

ACTIVITY 7

Find the gradient of the curve $f(x) = 2x^2 - 5x + 6$ at any point x.

ANSWERS ON PAGE 54

Now try another example.

ACTIVITY 8

What is the slope of $f(x) = 5x^2 - 4x + 7$ at any point along the curve? Find the slope at the point $x = 1$.

ANSWERS ON PAGE 55

The derivative of a function

There is nothing new to learn in this section. It is more like a conclusion. The gradient of a curve at any point is called the derivative of the function. We have already seen in the above section that the derivative of x^2 is $2x$. We also saw that the derivative $2x^2 - 5x + 6$ is $4x - 5$.

Let us try and work out some general formulas for the derivatives of some common functions. We can then memorise them instead of having to calculate limits each time.

a) The derivative of $f(x) = ax$, where a is any real number.

$$f(x+h) = a(x+h) = ax + ah$$

$$f(x+h) - f(x) = ax + ah - ax + ah$$

$$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{ah}{h}$$

$$= \lim_{h \to 0} a$$

$$= a$$ (The limit of a constant is just that, constant)

We write $f'(x) = a$.

We can then summarise our findings for any function of the form $f(x) = ax$.

The derivative of $f(x) = ax$ is equal to a
written $f'(x) = a$

ACTIVITY 9

Write down the derivatives of the following:

1. $3x$

2. $6x$

3. $-5x$

4. $-22x$

ANSWERS ON PAGE 55

b) The derivative of $f(x) = ax^2$, where a is any real number

$$f(x+h) = a(x+h)^2 = a(x^2 + 2xh + h^2) = ax^2 + 2axh + ah^2$$
$$f(x+h) - f(x) = 2axh + ah^2 = (2ax + ah)h$$
$$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(2ax + ah)h}{h}$$
$$= \lim_{h \to 0} (2ax + h)$$
$$= 2ax$$

The derivative of $f(x) = ax^2$ is equal to $2ax$ written as
$f'(x) = 2ax$

ACTIVITY 10

Write down the derivatives of the following:

1. $3x^2$
2. $-3x^2$
3. $15x^2$
4. $-34x^2$

ANSWERS ON PAGE 55

c) The derivative of $f(x) = ax^2 + bx + c$, where a, b and c are any real numbers.

How about doing this one in an activity?

ACTIVITY 11

Find the derivative of the function $f(x) = ax^2 + bx + c$.

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ACTIVITY 12

Write down the derivatives of the following:

1. $3x^2 + 4x - 6$
2. $-2x^2 + 4x + 10$
3. $-4x^2 - 7x + 5$
4. $5x^2 - 13x + 24$

ANSWERS ON PAGE 56

We now give a formal definition of the derivative of a function:

Definition:

The derivative of the function $f(x)$ is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$, if the limit exists.

- We write $f'(x) = \lim_{h \to 0} \frac{f(x+a) - f(x)}{h}$
- When we calculate a derivative using the above definition we say we are using 'first principals'.

ACTIVITY 13

Calculate the derivatives of the following functions from first principles.

1. $g(x) = x^3$
2. $p(x) = \frac{1}{x}$
3. $c(x) = k$ (k is a constant real number)

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Lets summarise the derivatives we have calculated so far:

$f(x)$	$f'(x)$
x^3	$3x^2 = 3x^{3-1}$
x^2	$2x^1 = 2x^{2-1}$
$x = x^1$	$1 = 1.x^0 = 1.x^{1-1}$
$k = kx^0$	$0 = 0.kx^{-1} = 0.kx^{0-1}$
$\frac{1}{x} = x^1$	$\frac{-1}{x^2} = -1.x^{-2} = -1x^{-1-1}$

ACTIVITY 14

By studying the above table write down the derivatives of

1. $f(x) = x^4$
2. $f(x) = x^5$
3. $f(x) = \frac{1}{x^2}$ $(= x^{-2})$
4. $f(x) = x^n$ where n is any integer

ANSWERS ON PAGE 57

Rules for derivatives

Finding derivatives from the first principle can be very tedious and, as we saw in the previous activity, quite of ten there is a simple rule for writing down a derivative. We list some of the rules below.

Rule 1 The derivative of a constant is zero.
i.e. If $f(x) = k$ (constant) then $f'(x) = 0$

Rule 2 If $f(x) = x^n$ then $f'(x) = nx^{n-1}$. Although we have only used integer values for *n* this rule actually works for any real number value of *n*, too.

Rule 3 If $f(x) = a.g(x)$ where *a* is a real number then $f'(x) = ag^1(x)$.
In other words, the derivative of a constant times a function is equal to the constant times the derivative of the function.
E.g. If $f(x) = 5x^3$ then $f'(x) = 5(3x^2) = 15x^2$

Rule 4 If $f(x) = g(x) \pm h(x)$ then $f'(x) = g'(x) \pm h'(x)$
The derivative of the sum or difference of two functions is equal to the sum or difference of the derivatives of the functions.
E.g. if $f(x) = x^3 + 2x^4 - 3x^5$
Then $f'(x) = [3x^2] + [2.4x^3] - [3.5x^4]$

$$= 3x^2 + 8x^3 - 15x^4$$

N.B. The derivative of a product or quotient of two functions is **not** the product or quotient if the derivatives of the functions.

e.g If $f(x) = x^3.x^4$ then $f'(x) \neq (3x^2)(4x^3) = 12x^5$ because $f(x) = x^7$ so $f'(x) = 7x^6$.

ACTIVITY 15

Write down the derivatives of the following functions:

1. $f(x) = x^3 - x^2 + x + 5$

2. $g(x) = 3x^2 - 4x + 6$

3. $h(x) = -\frac{1}{2}x^4 + 7x^2 - \sqrt{2} + \frac{3}{x^2} - \frac{10}{x^4}$

4. $i(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x}$

5. $j(x) = (x^2 + 1)(x - 3)$

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CHECKLIST

Are you able to:

- find the gradient between two points on a curve
- differentiate functions
- solve problems by means of the Calculus
- investigate average gradients
- find derivatives using first principles and by inspection

SELF-CHECK EXERCISE

1. Find the average gradient of $f(x)=x^3$ between $x = 1$ and $x = 3$.

2. Use first principles to calculate the derivatives of the following functions:

   a) $(x)=3x^2+1$ b) $q(x)=\dfrac{2}{x}$

3. Write down derivatives for the following (use the rules for differentiation):

   a) $f(x)=7x$

   b) $f(x)=x^2+4$

   c) $f(x)=3x^2-6$

   d) $f(x)=-x^2+6x-4$

   e) $f(x)=2x^2+3x+5$

   f) $f(x)=\dfrac{x^2-5x+6}{x^2-3x}$

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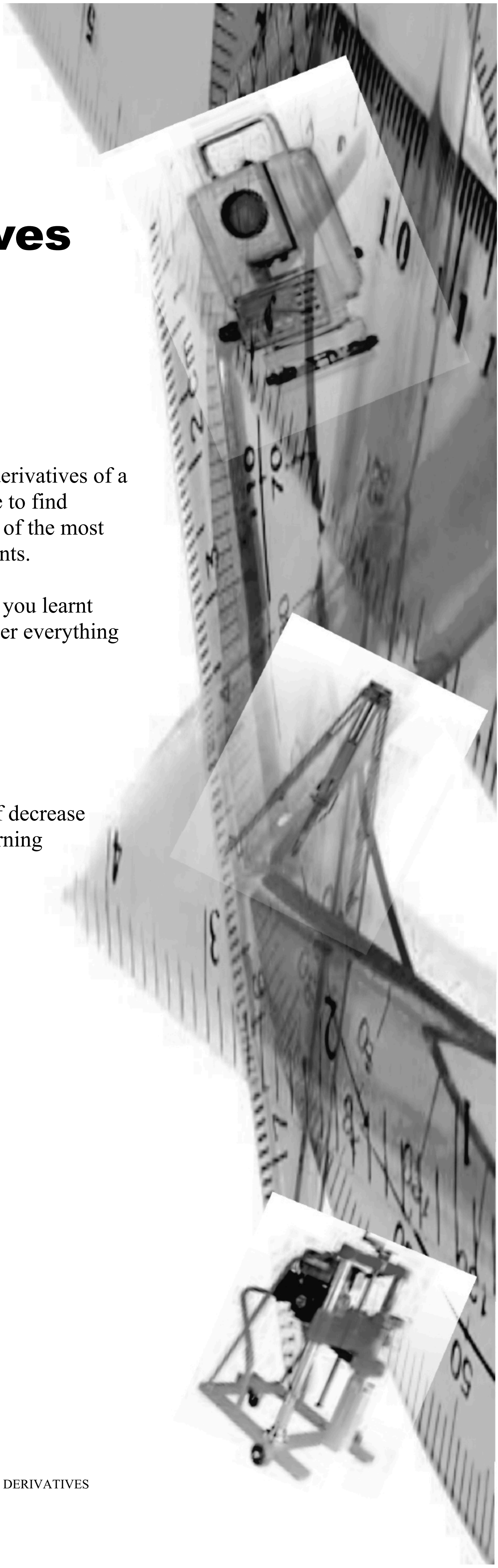
LESSON 3

Interpreting derivatives



About this lesson

In the previous lesson you were shown how to find the derivatives of a few functions. In this lesson you will use this knowledge to find equations of tangents, to analyse functions, and find one of the most important points in the analysis of functions: turning points.

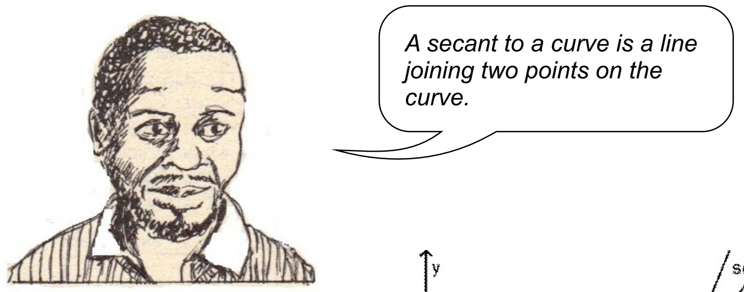
For this lesson you will need to recall most of the things you learnt about quadratic functions. You will also need to remember everything you learnt in the last lesson.

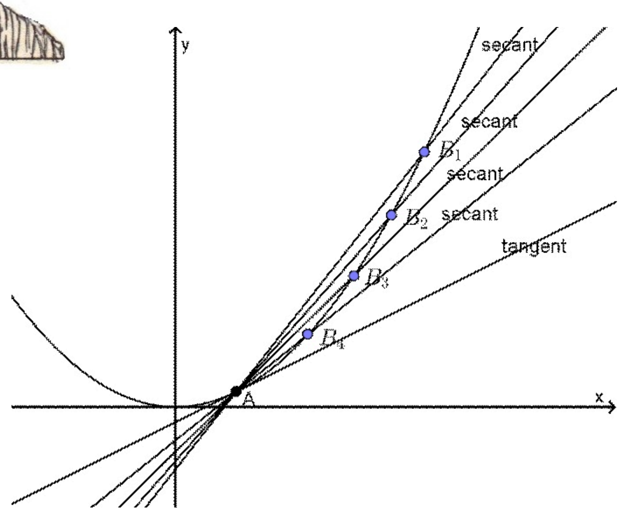
In this lesson you will:

- find the equation of tangents to curves
- use derivatives to show when functions increase of decrease
- use derivatives to find maximum and minimum turning points of functions.

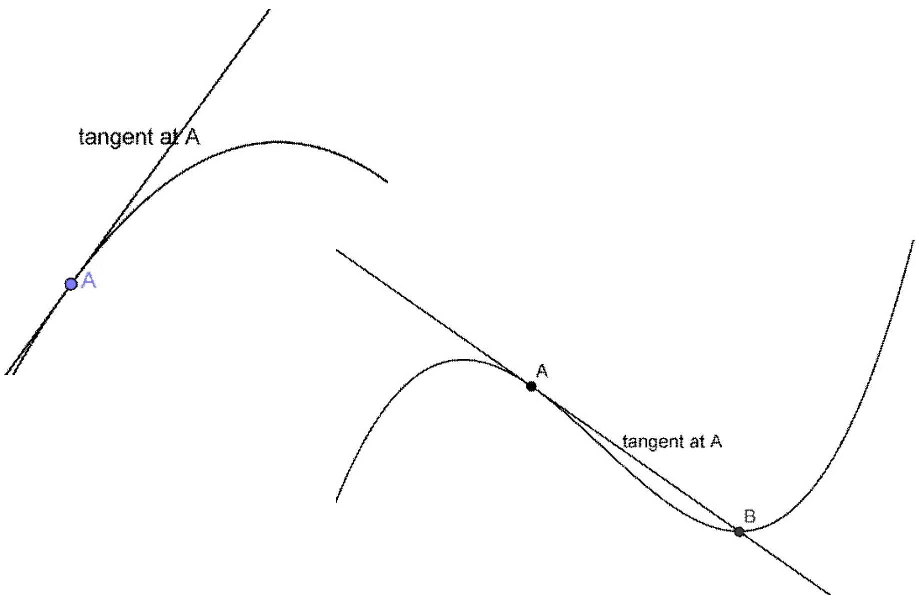
Tangents to curves

In Lesson 2 we touched on the concept of a tangent to a curve at a point on the curve as being the limiting position of secants through the point.





Intuitively, we would want a tangent to a curve at a point A on the curve to pass through (touch) the curve at only that point, at least in the vicinity of A. Further from A it may possibly cut the curve again.



It seems reasonable that the gradient of the tangent to a curve at A should have the same gradient (or slope) as the curve itself at the point A.

Definition: The tangent to the curve $y = f(x)$ at the point A $(x_1; y_1)$ on the curve is the straight line through A with gradient $m = f'(x_1)$.

Example:

To find the equation of the tangent to $y = x^2$ at the point A(2; 4) on the curve we need to find the gradient (i.e. derivative) of $y = x^2$ at $x = 2$.

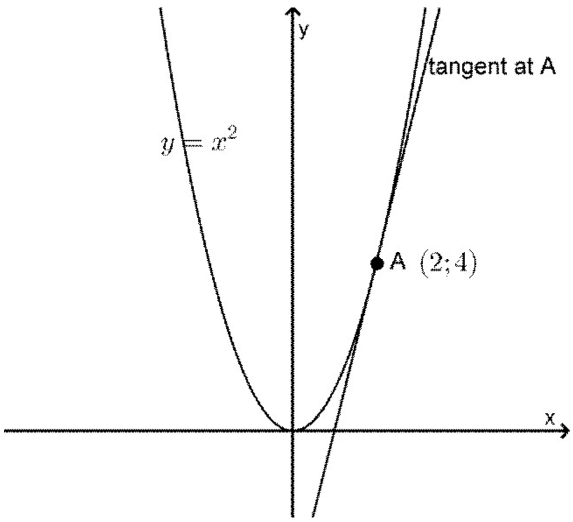
The derivative of x^2 is $2x$ so the gradient / slope / derivative of $y = x^2$ at $x = 2$ is $2(2) = 4$ i.e. the gradient of the tangent to $y = x^2$ at A is also 4.

Remember, the equation of a line with gradient m and passing through $(x_1; y_1)$ is $y - y_1 = m(x - x_1)$.

So the equation of this tangent with gradient $m = 4$ and passing through A (2;4)

is $y - 4 = 4(x - 2)$

i.e. $y = 4x - 4$.



ACTIVITY 1

Find the equations of the tangents to the curve $y = f(x)$ at the given points.

1. $f(x) = x^2 + 2$ at $(-1; 3)$
2. $f(x) = x^3$ at $(x = 2)$
3. $f(x) = 4 - 2x$ at $x = 0$

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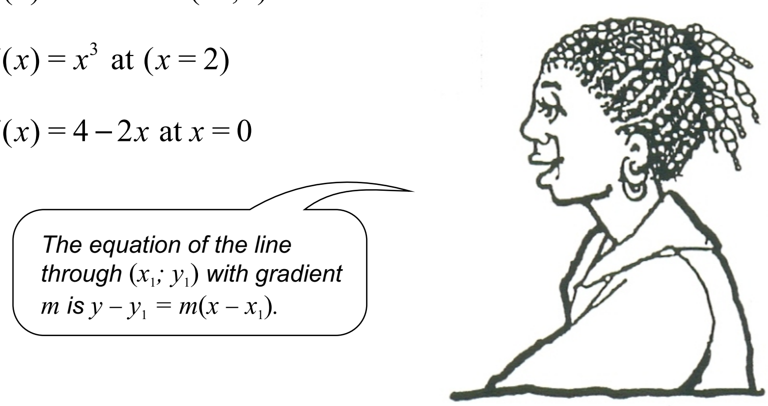

Note

- The tangent to a straight line must (obviously) be the line itself at all points on the line.
- The derivative of a linear function is a constant since the gradient of the line is the (same) constant.

ACTIVITY 2

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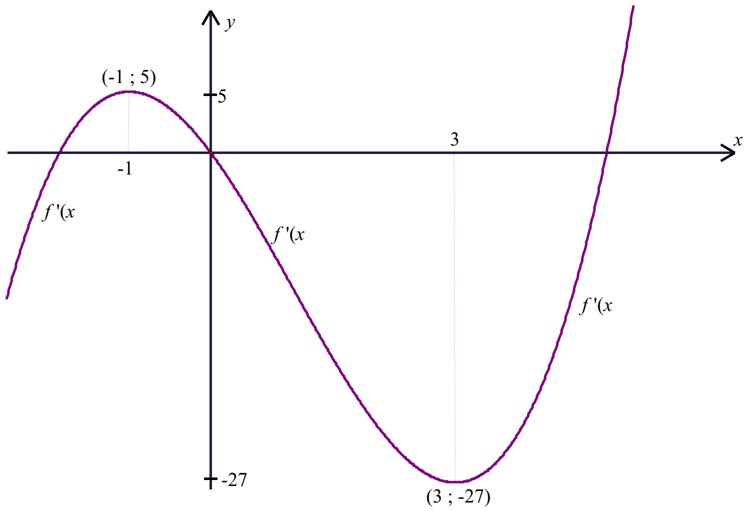
Find the point(s) on $y = f(x) = x^\wedge$ at which the gradient to the curve is 12. Find the equation of the tangents at the point(s).

Increasing, decreasing and stationary points

Consider the graph $y = f(x)$ below



We use $+$ to indicate where the gradient is positive
0 to indicate where the gradient is zero
$-$ to indicate where the gradient is negative

Remember, we describe a function as increasing or decreasing from the left to the right (increasing x).

We see that this graph is

increasing (i.e. y is increasing) and $f'(x) > 0$ for $x < a$
constant (horizontal tangent) and $f'(x) = 0$ at $x = a$
decreasing (i.e. is decreasing) and $f'(x) < 0$ for $a < x < b$
constant (horizontal tangent) and $f'(x) = 0$ at $x = b$
increasing and $f'(x) > 0$ for $b < x < c$
constant at $x = c$
decreasing and $f'(x) < 0$ for $c < x < d$
constant and $f'(x) = 0$ at $x = d$
increasing and $f'(x) > 0$ for $x > d$

We say the function f

i) is increasing whenever $f'(x) > 0$
ii) is decreasing whenever $f'(x) < 0$
iii) has a stationary point whenever $f'(x) = 0$

ACTIVITY 3

Find the stationary points and where *f* is increasing or decreasing in each of the following cases:

1. $f(x) = x^2$

2. $f(x) = 8 - 2x - x^2$

3. $f(x) = x^3 - 3x^2 - 9x$

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Maxima and minima

A stationary point of a function *f* (i.e. where $f'(x) = 0$) is called a turning point if *f* changes from either increasing or decreasing or from decreasing to increasing at the point.

In the graph in the last activity:

- $(-1;5)$ is a **maximum** turning point because $f'(-1) = 0$ and *f* changes from increasing to decreasing at $(-1;5)$.

- $(3;-27)$ is a **minimum** turning point because $f'(3) = 0$ and *f* changes from decreasing to increasing at $(3;-27)$.

ACTIVITY 4

1. Find all the turning points of $f(x) = x^3 - 12x + 12$ and determine whether each is a maximum or a minimum turning point.

2. Show that $g(x) = x^3$ has a stationary point which is not a turning point.

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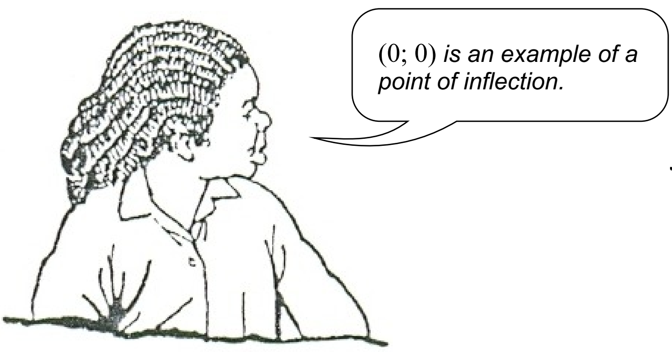
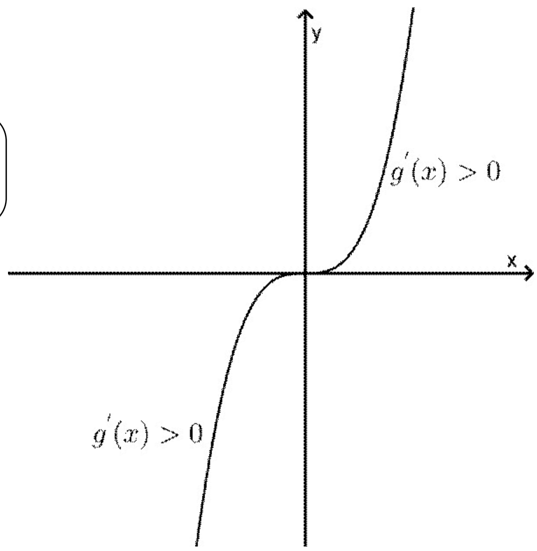
Although (0; 0) is a stationary point, *g* increases up to $x = 0$, stops for an instant, then continues after $x = 0$.
(0; 0) is a stationary point but not a turning point.

Here is the graph $y = x^{\wedge}$:





ACTIVITY 5

1. Use the methods discussed in this lesson to find the turning points of the following parabolas. Sketch the parabolas.

   a) $y = f(x) = x^2 - 2x - 3$

   b) $y = g(x) = -x^2 + 4x + 12$

2. Show that the turning point of the parabola $y = p(x) = ax^2 + bx + c$ occurs at $x = -\frac{b}{2a}$ .When is the turning point a maximum or minimum?

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Summary

In this lesson we have learnt that:

- the equation of the tangent to $y = f(x)$ at A(x; y) on the curve is given by $y - y_1 = m(x - x_1)$ where $m = f'(x_1)$
- a function f:
  - is increasing when $f'(x) > 0$,
  - decreasing when $f'(x) < 0$
  - has a stationary point when $f'(x) = 0$.
- a stationary point is:
  - a maximum turning point if the function changes from increasing to decreasing at the point.
  - a point of inflection if the function remains increasing or decreasing either side of the point.
  - a minimum turning point if the function changes from decreasing to increasing at the point

CHECKLIST

Are you able to:

- find equations of tangents
- use derivatives to determine when functions increase or decrease
- find maximum and minimum turning points of functions.

SELF-CHECK EXERCISE

1. Find the equation of the tangent to the curve $y = h(x) = \frac{1}{x}$ at the point (1;1).

2. Find and classify the turning points of the function defined by $f(x) = 2x^3 - 3x^2 - 36x + 10$.

3. Find the equation of the tangent to the graph $y = x^2 - 2x + 5$ at its turning point.

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LESSON 4

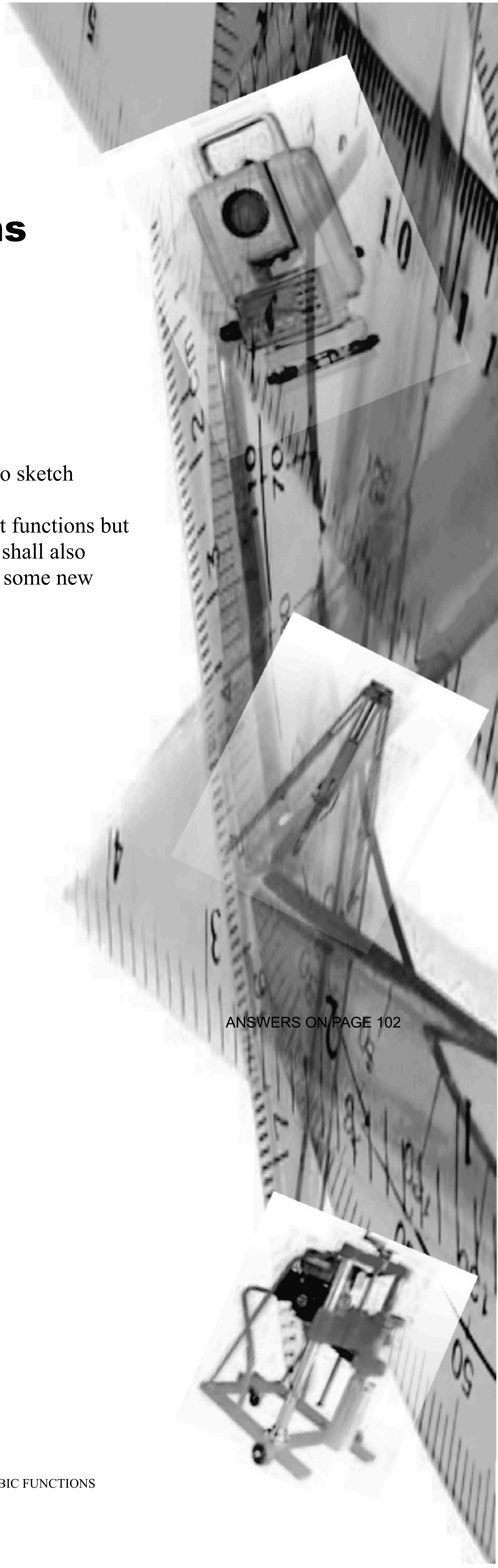
Sketching cubic functions

About this lesson

In this lesson we shall apply what we learnt in lesson 3 to sketch graphs of cubic functions, that is functions of the form $f(x) = ax^3 + bx^2 + cx + d$. Our method will work for most functions but only cubic functions will be asked in examinations. We shall also revise how to factorise cubic polynomials and introduce some new notation for derivatives.

In this lesson you will:

- use the notation $\dfrac{dy}{dx}$ for derivatives.
- sketch cubic functions.
- interpret graphs of cubic functions.



ANSWERS ON PAGE 102

Revision of concepts

ACTIVITY 1

For $f(x) = x^3 - 5x^2 + 7x - 2$:

1. Calculate $f(1)$, $f(-1)$, $f(2)$, $f(-2)$ and $f(0)$.
2. Find the quotient and remainder when $f(x)$ is divided by $x - 2$.
3. Find all the roots (solutions) of the equation $f(x) = 0$.
4. Find all the stationary points of $y = f(x)$.
5. Decide whether the stationary points are maximum or minimum turning points.

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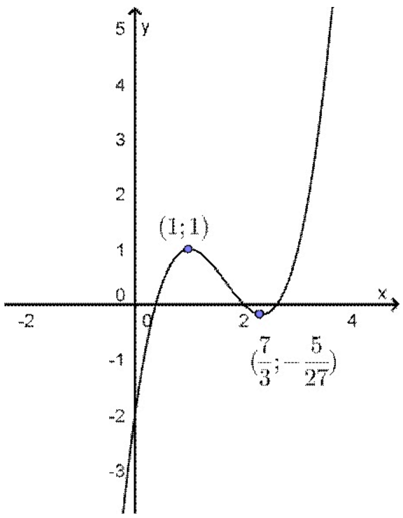
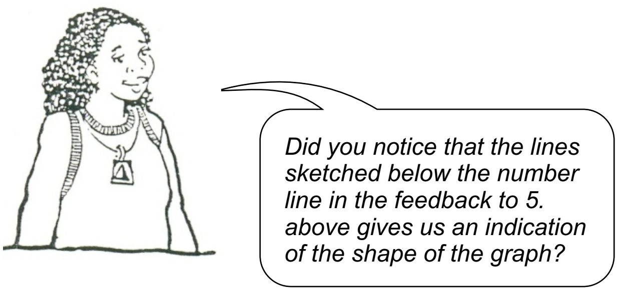
Note:
Although the above activity was meant as a revision exercise, can you see that we actually have enough information to make a rough sketch of the cubic function $y = x^3 - 5x^2 + 7x - 2$?

Here it is (not to scale):





Alternative notations for derivatives

Up until now we have always indicated the derivative of $f(x)$ by $f'(x)$, thanks to a French mathematician by the name of Lagrange. One of the founders of calculus, Leibuitz, used the notation $\frac{dy}{dx}$, where $y = f(x)$, which is still widely used today.

Newton developed the Calculus at the same time as Leibuitz used the word fluxion for derivative and wrote $\dot{y}$, which is still used in Physics and applied Mathematics.

Thus if $y = f(x)$ then $f'(x) = \frac{dy}{dx} = \dot{y}(x) = \frac{df(x)}{dx}$.

ACTIVITY 2

Find $\frac{dy}{dx}$ in each of the following cases.

1. $y = 3x^3 - 6x^2 + 5x + 2$
2. $y = f(x)$ where $f(x) = x^2 + \frac{1}{x^2}$

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The notation $\frac{dy}{D_x}$ will prove very useful for curve sketching and we shall use it often. Note that $\frac{dy}{D_x}$ is one symbol and should not be thought of as $dy \div dx$.

You may see the notation D_x in some books; for example $D_x[x^2 + 3x - 5] = 2x + 3$. We shall not use this notation.

Sketching cubic functions

To sketch the graph of a cubic function $y = f(x)$ follow the steps below: (have a calculator handy).

Step 1 Find the intercepts with the axes
For the x –intercept(s), put $y = 0$ and solve.
For the y –intercept, put $x = 0$ (i.e. find $f(0)$)
(To solve $y = 0$ you may need to use the factor theorem and the quadratic formula.)

Step 2 Find $\frac{dy}{dx}$ (i.e. $f'(x)$)

Step 3 Find any stationary points by solving $\frac{dy}{dx} = 0$.

Step 4 Use a table of signs (number line) to find where the graph is increasing or decreasing and classify the stationary points as maximum or minimum turning points or points of inflection.

Step 5 Plot the points using a set of axes (not necessarily to scale) and join them to make a rough sketch of the curve.

N.B. The curve must be smooth with no sharp points!

ACTIVITY 3

Use the steps above (as required) to draw the graphs of the following cubic functions.

1. $y = x^3 - 9x$
2. $y = x^3 - 3x^2 + 2$
3. $y = 8 - x^3$

ANSWERS ON PAGE 65

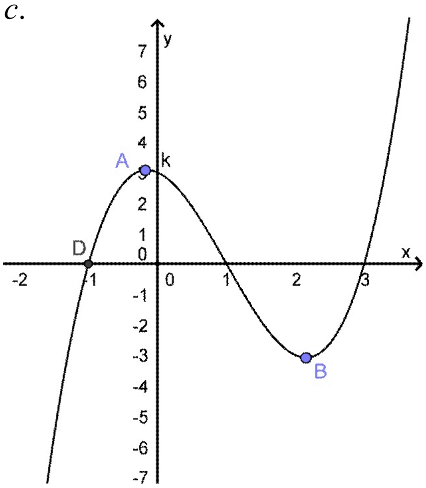
Interpreting graphs of cubic functions

As you did with quadratic functions, you used to be able to look at the graph of a cubic function and obtain information from it. There is nothing new in this section - we shall simply apply the techniques we have learnt thus far.

ACTIVITY 4

Below is a sketch of $y = f(x)$ where $f(x) = x^3 + ax^2 + bx + c$.

1. Calculate the values of a, b and c.
2. Determine the co-ordinates of k, the y –intercept, and the length of OK.
3. If A and B are the turning points of $y = f(x)$, determine the x –coordinates of A and B.
4. Find the equation of the tangent to $y = f(x)$ at $x = 2$.
5. Does the tangent in 4 cut the graph again? If so, where?



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Note:

1. You are unlikely to be asked a question like 5. above, so if you found it tricky omit it.
2. $x = 2$ gives us the point where the tangent touches the graph which, thinking back to average gradient, could be thought of as 'occurring twice' and hence the square factor $(x - 2)^2$. This always happens at a point where the tangent touches the graph of a cubic function.

CHECKLIST

Are you able to:

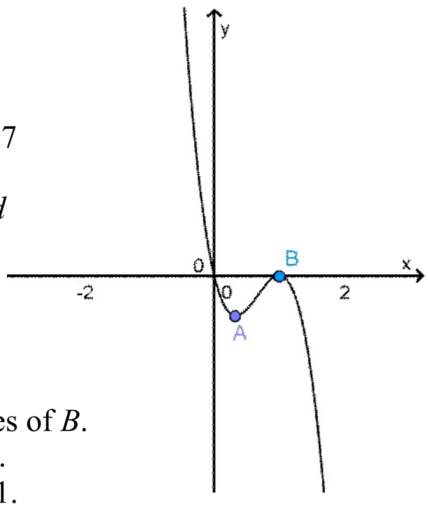
- use the notation $\frac{dy}{dx}$ for derivatives.
- sketch cubic functions.
- interpret graphs of cubic functions.

SELF-CHECK EXERCISE

1. Use the methods learnt in this lesson to sketch the graph of the following functions:

   a) $y = f(x) = 2x^2 - 8x + 6$
   b) $y = g(x) = x^3 - 3x^2 + 4$
   c) $y = h(x) = -x^3 + 12x$
   d) $y = i(x) = -x^3 - 3x^2 - 3x + 7$

2. The graph $y = -x^3 + bx^2 + cx + d$ is shown on the right. It passes through the origin and A and B are turning points. B is also an x –intercept.



   a) Write down the coordinates of B.
   b) Write down the value of d.
   c) Show that $b = 2$ and $c = -1$.
   d) Find the coordinates of A.

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LESSON 5

Application of the Calculus

About this lesson

In this lesson, you are going to use calculus to solve problems that are in other fields like biology, social science, management, economics and engineering. For this lesson, you will need almost all the information you have learnt from the previous unit.

In this lesson, you will:

- discover various areas where calculus is used
- solve practical problems using calculus

Calculus in biological sciences

Example

A biologist estimates that the number of bacteria present in a culture of bacteria at any time *t* is given by:

$$B(t) = 1000 + 50t - 5t^2$$

where B(*t*) is the number of bacteria, in millions, present at time *t*, measured in hours.

Find the rate of change of the number of bacteria with respect to the time for the following values of *t*,

a) $t = 2$
b) $t = 3$
c) $t = 4$
d) $t = 5$
e) $t = 6$
f) $t = 8$

Solution

The function B(*t*) is a quadratic function. The rate of change at any instant is the derivative of B(*t*). Therefore, we must first find $B'(t) = 50 - 10t$.

We can now find the rate of change of the number of bacteria at the given times.

(a) $B'(2) = 50 - 10(2) = 30$
(b) $B'(3) = 50 - 10(3) = 20$
(c) $B'(4) = 50 - 10(4) = 10$
(d) $B'(5) = 50 - 10(5) = 0$
(e) $B'(6) = 50 - 10(6) = -10$
(f) $B'(8) = 50 - 10(8) = -30$

At first, the population of the bacteria is increasing (positive derivative) although at diminishing rates.

At 5 hours, the rate of change is zero. This is the point where the situation changes. At 5 hours there is no increase at all. After 5 hours, the population starts decreasing, because the bacteria are dying faster.

ACTIVITY 1

Insulin affects the glucose, or blood sugar, level of some diabetics, according to the function

$$G(x) = -0,2x^2 + 450$$

where $G(x)$ is the blood sugar level one hour after x units of insulin are injected. (This mathematical model is only approximate. It is valid for low amounts of insulin.)

1. Find $G(0)$

2. Find $G(25)$

3. Find $\frac{dG}{dx}$ for $x = 10$ and $x = 25$

Interpret your answers.

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ACTIVITY 2

Some people suffer from narrowing of the blood vessels associated with various health problems. Problems with circulation of the blood vessels can be severe, and may in special circumstances lead to heart attack or stroke.

Medicines that cause blood vessels to dilate (expand) are called vasodilators. These may sometimes help the symptoms.

A short length of blood vessel is like a cylinder. The volume of a cylinder is given by:

$$V = \bullet\, r^2 h$$

Suppose an experiment is set up to measure the volume of blood in a blood vessel of length 80mm, where r is the radius of the blood vessel and h the length.

A drug which causes blood vessels to expand (increase in diameter and therefore radius) is administrated.

1. The volume of blood in the blood vessels changes as radius changes. Find the rate at which the volume of blood changes with radius i.e. find a formula for $\frac{dv}{dr}$.

2. Evaluate $\frac{dv}{dr}$ for the following values of r.

   a) 4mm
   b) 6mm
   c) 8mm

ANSWERS ON PAGE 71

Motion and Calculus

Calculus is a very practical part of mathematics. We apply it in many fields. In most cases at school it is applied in analysing motion. Here is basic information you need to know:

Time: time is usually given in seconds or hours. The various functions of motion depend on time. Therefore time is usually the independent variable. The symbol used is t.

Displacement: displacement unlike what many people think, is not simply distance covered over a given period of time. Displacement gives distance travelled in a straight line in a given time. The symbol used is usually $s(t)$.

Velocity: velocity is the speed at which an object moves in a straight line. The symbol used is usually $v(t)$. To get velocity, we take the derivative of displacement with respect to time, that is $v(t) = s'(t)$.

Acceleration: acceleration is the rate of change of velocity. The symbol used is usually $a(t)$. Acceleration is expressed as the derivative of velocity with respect to time, i.e. $a(t) = v'(t)$.

How are problems usually structured? Let us find out through an activity.

ACTIVITY 3

The displacement of a particle in motion is given by the equation $s(t) = 3t^2$, where s is in metres and t is in seconds. Find the following.

1. What is the initial displacement of the particle?
2. How far is the particle from its origin after 3 seconds?
3. What is the velocity of the particle at any given time?
4. What is the velocity of the particle at the time $t = 3$?

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It is necessary to explain your answer in words. It is also very important to state the units you have used. Try the following activity.

ACTIVITY 4

A shell is fired vertically upwards from the ground. The height of the shell is given by the equation

$$s(t) = 300t - 5t^2$$

where s is the height in meters and t is the time in seconds.

1. What is the initial velocity of the shell (i.e. when $t = 0$)?

2. What is the greatest height reached?

3. How long will the shell take before it hits the ground?

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ACTIVITY 5

The velocity, v m/s, of a body moving in a straight line is given by

$$v(t) = t^2 - t$$

1. Find the acceleration of the body after 3 seconds.

2. Find when the acceleration is zero.

3. What is happening to the acceleration at the time $t = 0s$?

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ACTIVITY 6

If the displacement of a particle after t seconds is given by $s(t) = 9 + 12t - 2t^2$ metres, find the velocity and acceleration of the particle after 3 seconds. Explain what is happening.

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Various Problems

In this section we will deal with number and area problems.

Example 1
The sum of the two positive numbers is 24. What is their maximum product?

Solution
Let the numbers be x and y. $x + y = 24$

Therefore, $y = 24 - x$

The aim is to maximize the product, $P = xy$.

$$P = x(24 - x) = 24x - x^2$$
$$P'(x) = 24 - 2x$$

(Parabola with a maximum turning point)

Maximum $P'(x) = 0$

$$24 - 2x = 0$$
$$x = 12$$

Therefore, $y = 12$

Maximum product $= 12 \times 12 = 144$

Example 2
A rectangle has a perimeter of 24 metres. Find the maximum possible area.

Solution
Let the length be l and the width be w.

$$\text{Therefore, perimeter,} \quad P = 2(l + w)$$
$$2(l + w) = 24$$
$$\text{Therefore, } l = 12 - w$$
$$\text{Area of rectangle} = l.w = (12 - w)w$$
$$A(w) = 12w - w^2$$
$$A'(w) = 12 - 2w$$
$$\text{Maximum area } A'(w) = 0$$
$$12 - 2w = 0$$
$$w = 6$$
$$\text{Therefore, } l = 6$$
$$\text{Maximum area} = l.w = 36 \text{ m}^2$$

To get maximum area, the rectangle must be a square.

In both questions you will note the following:

- we write an equation on the basis of given information
- we express one of the variables in terms of the other. This helps to have a function that has only one variable.
- we write an equation for what needs to be maximised or minimised.
- we find the value of the variable where the function is at maximum or minimum. This is when the derivative is zero.

Now try the following activity.

ACTIVITY 7

ANSWERS ON PAGE 74

The sum of two numbers is 20. Find the numbers if the sum of their squares is a minimum.

ACTIVITY 8

A cylinder is to be made such that the radius and its height is 6 m. Find the maximum volume of the cylinder.
(Remember: volume of cylinder = $V = \pi\ r^2h$.)

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ACTIVITY 9

A dressmaker making school uniforms has found that her costs are given by

$$C(x) = 2x$$

and her revenue (income from sales) by

$$R(x) = 6x - \frac{x^2}{1000}$$

where x is the number of items produced.

1. Find the number of items she can produce to get the highest revenue.

2. Find the number of items she can produce to get the highest profit.

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Summary

To get the maximum or minimum of any process or function $f(x)$ we get the derivative $f'(x)$ and look for values of x when $f'(x) = 0$.

You may have noted that we do not spend time proving whether the point is a maximum or minimum. This is because we know that in $ax^2 + bx + c$, if a is a negative, the critical point is a maximum point, and if a is positive, the critical point is a minimum point.

CHECKLIST

Are you able to:
- use calculus to solve problems
- find maxima and minima in practical problems

How about exercises to check whether you understand?

SELF-CHECK EXERCISE

1. The velocity of a body, in metres per second, is given as $v(t) = 4t - t^2$.
   a) When is the body momentarily at rest?
   b) Find the acceleration when the body is at rest.

2. A farmer wants to enclose sheep in a rectangular pen. If he has 600 metres of wire fence, find the greatest area he can enclose.

3. The total number of tourists visiting a holiday resort is found to be $T(x) = -x^2 + 50x + 150$ where x is the total rainfall in centimetres for the previous week.
   a) Find the rainfall that would produce the highest number of tourists.
   b) Find the maximum number of visitors.

4. In a factory where boxes (crates) are packed, their engineer found that the displacement of a crate sliding down a 50 metre groove is given by: $x = 10\text{it} - 0{,}5t^2$,
   Where x is in metres (m), and t is in seconds (s)
   a) Draw the graph of the displacement for values of t from 0 to 10.
   b) How far is a crate 5s, 7s?
   c) How long does a crate take to finish the slide?
   d) What is the velocity of the crate when $t = 0$ seconds, 1 second and 8seconds?

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Feedback to Activities

Lesson 1

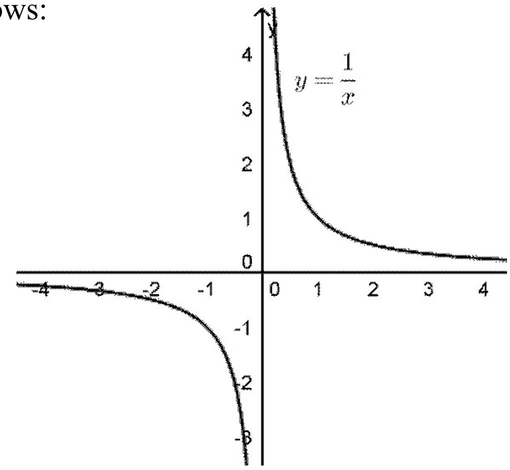
Activity 1

For the tables we shall use values of x in [–5, 5]. This means that we start from x = –5 and work out values of the functions until we come to $x = 5$, but the functions and their graphs exist for all x –values for which they are defined.

a) $f(x) = \dfrac{1}{x}$

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
$f(x)$	$-\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	–1	***	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

It is clear here that the function does not exist at $x = 0$. We draw the graph as follows:

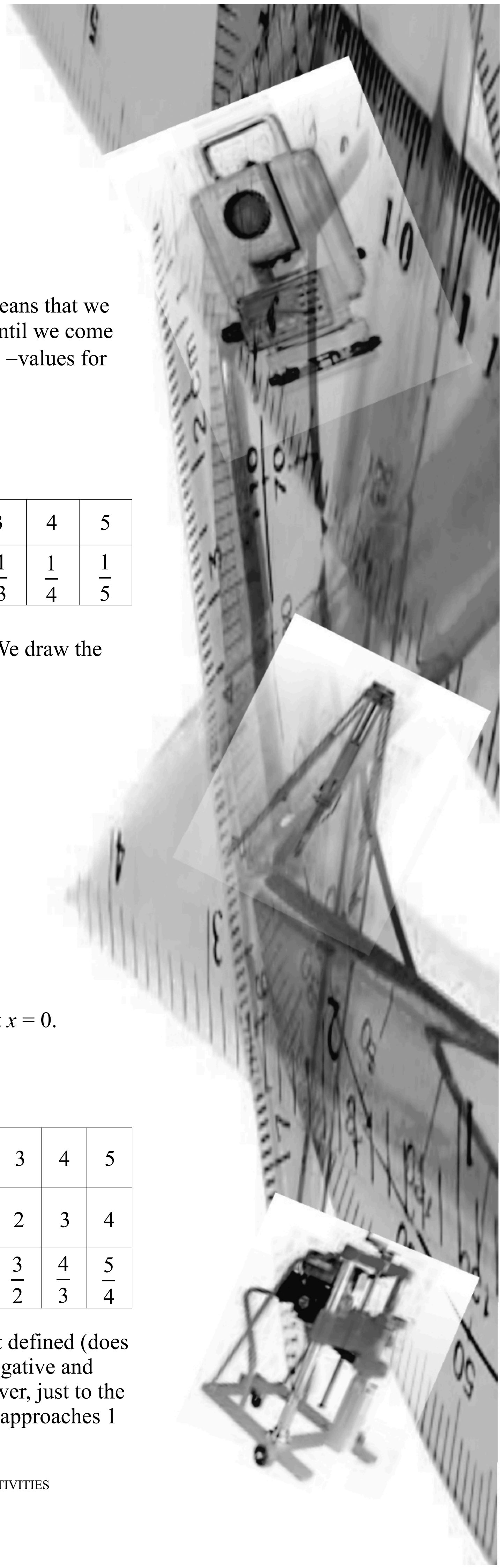


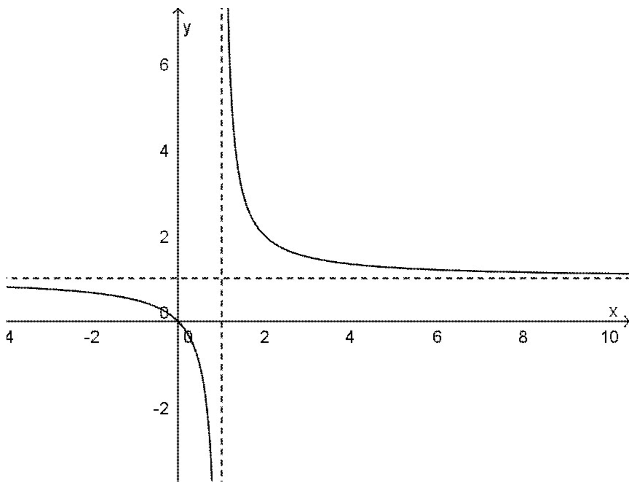
The function is not continuous (the graph is 'broken') at $x = 0$.

b) $f(x) = \dfrac{x}{x-1}$

x	–5	–4	–3	–2	–1	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3	4	5
$x-1$	–6	–5	–4	–3	–2	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3	4
$f(x)$	$\frac{5}{6}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	0	–1	***	3	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$

Here we have a problem when $x = 1$. The function is not defined (does not exist) at this point. Just to the left of $x = 1$, $f(x)$ is negative and decreases rapidly as x approaches 1 from the left. However, just to the right of $x = 1$, $f(x)$ is positive and increases rapidly as x approaches 1 from the right. (See the graph on the next page.)





Activity 2

1. $x = 2$ (i.e. when the denominator equals 0)

2. No. If we try we get $f(2) = \frac{4-2-2}{2-2} = \frac{0}{0}$ which cannot be evaluated (Note: $\frac{0}{0} \neq 1$).

3.

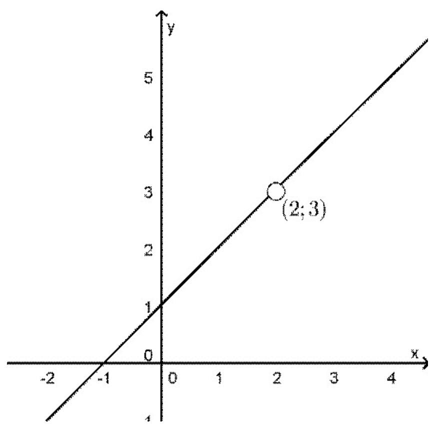
x	1,9	1,99	1,999	1,9999	2	2,0001	2,001	2,01	2,1
y	2,9	2,99	2,999	2,9999	?	3,0001	3,001	3,01	3,1

4. a) and b): $f(x)$ appears to approach 3 (which helps confirm that we should not think of $\frac{0}{0}$ as 1).

5. By plotting points or noticing that

$$f(x) = \frac{(x-2)(2+1)}{x-2} = x+1 \quad \text{when } x \neq 2$$

we get:



Notice that in this example there is no asymptote. The function is not continuous at $x = 2$ but the y –value 3 is approached in both cases this time as x approaches 2 from either the left or the right. We indicate the point (2; 3) with a small circle to show that it does not lie on the graph.

Activity 3

You should understand that the function approaches the value 4 as x approaches 2, both from the right, and from the left. This question is about showing that.

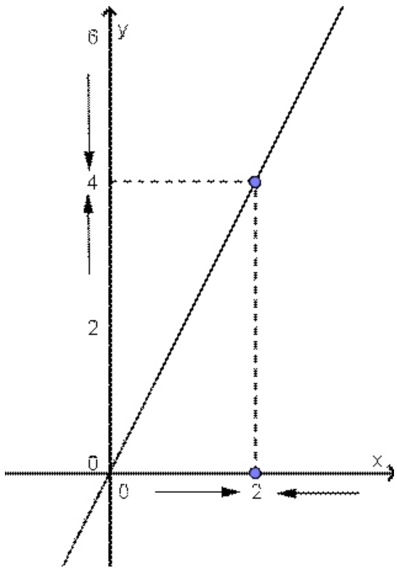
This can be done with the use of a calculator. Draw tables, starting from x = 1,95 towards x = 2,00, and from x = 2,05 going backwards towards x = 2,00. From both sides, stop at 1,99 and at 2,01 respectively. The tables could be as follows:

x	1,95	1,96	1,97	1,98	1,99
$g(x)$	3,90	3,92	3,94	3,96	3,98

x	2,05	2,04	2,03	2,02	2,01
$g(x)$	4,10	4,08	4,06	4,04	4,02

The tables show that the values fo the function approach the same value from both sides as the variable x approaches 2.

This is going to be clearer if we show it in a diagram.



In this example,
$g(2)=4$ and $g(x)\to 4$
as $x\to 2$, i.e. $\lim_{x\to 2} g(x)=4$
The function is continuous at $x=2$

Activity 4

The limits of the functions given here can be found by substituting the value of x into the formula given because each function is continuous (not broken) at the x –value.

1. The limit of $f(x) = 5x$ as x approaches 3 is equal to 15. We can write this mathematically as follows:

$$\lim_{x\to 3} f(x)=\lim_{x\to 3} 5x$$
$$=5(3)$$
$$=15$$

2. The limit of $3x - 1$ as x approaches 5 is equal to 14.

$$\lim_{x \to 5}(3x - 1) = 14$$

3. $$\lim_{x \to 5,2}(5x + 2) = 28$$

4. $$\lim_{x \to 3}(3x^2 + 4x - 2) = 37$$

Activity 5

x	–0,1	–0,05	–0,01	–0,001	–0,0005	–0,0001
$f(x)$	–10	–20	–100	–1 000	–2 000	–10 000

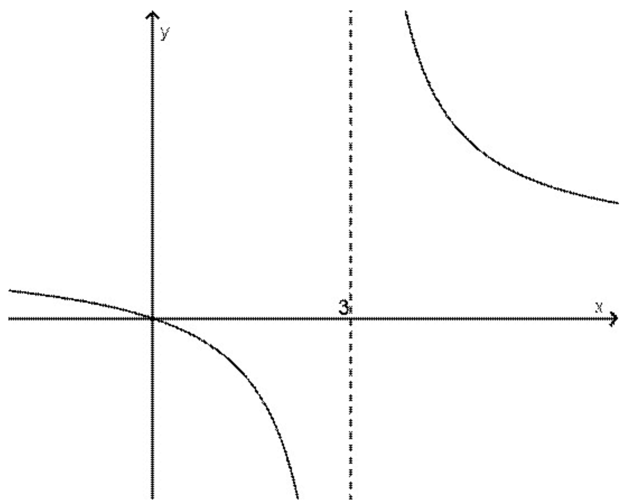
x	0,1	0,05	0,01	0,001	0,0005	0,0001
$f(x)$	10	20	100	1 000	2 000	10 000

Activity 6

Draw up a table of values for the following values of x:

–1; 0; 1; 2; 2,5; 2,75; 2,9; 3; 3,1; 3,25; 3,5; 4; 5

Here is the graph of the given function.



Activity 7

x	2,9	2,95	2,99	2,995
$x^2 - 9$	–0,59	–0,2975	–0,0599	–0,029975
$x - 3$	–0,1	–0,05	–0,01	–0,005
$f(x)$	5,9	5,95	5,99	5,995

x	3,1	3,05	3,01	3,005
$x^2 - 9$	0,61	0,3025	0,0601	0,030025
$x - 3$	0,1	0,05	0,01	0,005
$f(x)$	6,1	6,05	6,01	6,005

Activity 8

The method you used in Activity 8 is helpful. It is shorter, and it uses something that you already know, factorisation.

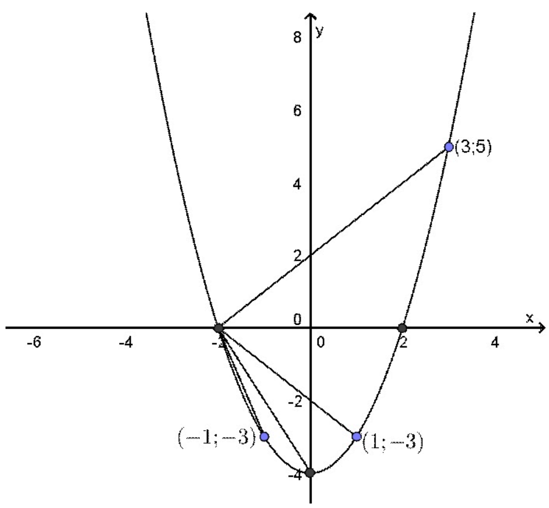
1. $$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x-2} &= \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}\\ &= \lim_{x\to 2}(x+2)\\ &= 4\end{aligned}$$

2. $$\begin{aligned}\lim_{x\to -3}\frac{x^2+7x+12}{x+3} &= \lim_{x\to -3}\frac{(x+3)(x+4)}{x+3}\\ &= \lim_{x\to -3}(x+4)\\ &= 1\end{aligned}$$

Lesson 2

Activity 1

a)



b) (i) $m = \dfrac{-3-0}{-1-(-2)} = -3 < 0$

(ii) $m = \dfrac{-4-0}{0-(-2)} = -2 < 0$

(iii) $m = \dfrac{-3-0}{1-(-2)} = -1 < 0$

(iv) $m = \dfrac{0-0}{2-(-2)} = 0$

(v) $m = \dfrac{5-0}{3-(-2)} = 1 > 0$

c) (i) Decreased on average ($m < 0$) at 3y units per x unit.

(ii) Decreased on average ($m < 0$) at 2y units per x unit.

(iii) Decreased on average ($m < 0$) at 1y unit per x unit.

(iv) Remained constant on average ($m = 0$)

(v) Increased on average ($m > 0$) at 1y unit per x unit.

d) (i) Actually decreases between (–2; 0) and (–1; –3). y decreased by 3 units when x increased by 1 unit.

(ii) Actually decreases between (–2; 0) and (0; –4) but not as fast as in (i). y decreased by 4 units when x increased by 2 units i.e. an average decrease of 2 units for an increase in x by 1unit.

(iii) Actually first decreased (between $x = -2$ and $x = 0$) but then actually increased; however, the average change was a slow average decrease of y by 1 unit for each increase in x by 1 unit.

(iv) Actually decreased between $x = -2$ and $x = 0$ but then increased between $x = 0$ and $x = 2$. The net change was 0. That is, the average rate of change was 0 y units for each increase in x by 1unit.

(v) Actually first decreased then increased. The net change was an average increase of y by 1 unit for each increase of x by 1unit.

Activity 2

Method 1

a. $h = x_2 - x_1 = 2 - 0 = 2$

$\therefore$ average gradient between the points on the graph $y = f(x) = 2x^2 - x + 1$ with $x_1 = 0$ and $x_2 = 2$ is:

Average gradient $= \frac{f(0+2)-f(0)}{2} = \frac{f(2)-f(0)}{2}$

$f(2) = 2(2)^2 - 2 + 1 = 7$ and $f(0) = 2(0)^2 - 0 + 1 = 1$

$\therefore$ average gradient $= \frac{7-1}{2} = 3$

b) $h = x_2 - x_1 = 1 - 0 = 1$

$\therefore$ average gradient $= \frac{f(0+1)-f(0)}{1} = \frac{f(1)-f(0)}{1}$

$f(1) = 2(1)^2 - 1 + 1 = 2$ and $f(0) = 1$

$\therefore$ average gradient $= \frac{f(1)-f(0)}{1} = \frac{2-1}{1} = 1$

Similarly (you do the arithmetic):

c) average gradient $= \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2}} = 0$

d) average gradient $= \frac{f(\frac{1}{4}) - f(0)}{\frac{1}{4}} = -\frac{1}{2}$

Method 2

First simplify $\frac{f(x_1+h)-f(x_1)}{h}$ when $x_1 = 0$

$f(x_1+h) = f(0+h) = f(h) = 2h^2 - h + 1$

$f(0) = 2(0)^2 - 0 + 1 = 1$

$$\frac{f(0+h)-f(0)}{h} = \frac{[2h^2 - h + 1] - [1]}{h} = \frac{2h^2 - h}{h} = 2h - 1$$

Now we simply calculate h in each case and substitute into $2h - 1$:

a) $h = 2_2 - x_1 = 2 - 0 = 2$
$\therefore$ average gradient = 2(2) – 1 = 3

b) $h = x_2 - x_1 = 1 - 0 = 1$
$\therefore$ average gradient = 2(1) – 1 = 1

c) $h = x_2 - x_1 = \frac{1}{2} - 0 = \frac{1}{2}$
$\therefore$ average gradient $= 2(\frac{1}{2}) - 1 = 0$

d) $h = x_2 - x_1 = \frac{1}{4} - 0 = \frac{1}{4}$

$\therefore$ average gradient = $2(\frac{1}{4}) - 1 = -\frac{1}{2}$

Activity 3

Let $h = x_C - x_A = x_C - 2$, then $x_C = 2 + h$ (notice that now $h < 0$)

$y_C = f(x_C) = f(2+h) = \frac{1}{2}(2+h)^2 = 2 + 2h + \frac{1}{2}h^2$ (as before!)

$\therefore$ average gradient of f between A (2; 2) and $C(x_C; y_C)$ is:

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_C - y_A}{x_C - x_A} \\ &= \frac{f(2+h) - f(2)}{h} \\ &= 2 + \frac{1}{2}h\end{aligned}$$

$C(x_C; y_C)$	$h = x_C - x_A$	$\frac{\Delta y}{\Delta x} = 2 + \frac{1}{2}h$
$C_1(1; f(1))$	$h = 1 - 2 = -1$	$\frac{\Delta y}{\Delta x} = 2 + \frac{1}{2}(-1) = 1,5$
$C_2(1,5; f(1,5))$	$h = 1 - 1,5 = -0,5$	$\frac{\Delta y}{\Delta x} = 2 + \frac{1}{2}(-0,5) = 1,75$
$C_3(1,9; f(1,9))$	$h = 1 - 1,9 = -0,1$	$\frac{\Delta y}{\Delta x} = 2 + \frac{1}{2}(-0,1) = 1,95$
$C_4(1,99; f(1,99))$	$h = 1 - 1,99 = -0,01$	$\frac{\Delta y}{\Delta x} = 2 + \frac{1}{2}(-0,01) = 1,995$
$C_5(1,999; f(1,999))$	$h = 1 - 1,999 = -0,001$	$\frac{\Delta y}{\Delta x} = 2 + \frac{1}{2}(-0,001) = 1,9995$
$C_6(1,9999; f(1,9999))$	$h = 1 - 1,9999 = -0,0001$	$\frac{\Delta y}{\Delta x} = 2 + \frac{1}{2}(-0,0001) = 1,99995$

Activity 4

Let us find the gradient using the idea of a limit, as in the previous example:

Please take care. Many people make mistakes with f (x + h). They write f (x), then add h to the expression. Be sure you are substituting correctly. Put (x + h) in the place of every x.

$$f(x) = x^2$$
$$f(x+h) = (x+h)^2$$

$$\text{If } x = 1$$
$$f(1) = 1^2$$
$$= 1$$

$$x + h = 1 + h$$
$$f(1+h) = (1+h)^2$$
$$= 1^2 + 2h + h^2$$

Therefore,

$$\text{gradient at } x = 1$$
$$= \lim_{h\to 0}\frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h\to 0}\frac{2h+h^2}{h}$$
$$= \lim_{h\to 0}(2+h)$$
$$= 2$$

Note: As h approaches 0, the value $2 + h$ approaches 2. Therefore, the gradient of the function $f(x) = x^2$ at point $x = 1$ is equal to 2.

Activity 5

$$f(x) = 3x^2 + 4$$
$$f(3) = 3\ (3)^2 + 4 = 27 + 4 = 31$$
$$f(x+h) = 3\ (x+h)^2 + 4 = 3\ (x^2 + 2xh + h^2) + 4$$
$$f(3+h) = 3\ (9 + 2\ (3)h + h^2) + 4$$
$$= 31 + 18h + 3h^2$$
$$f(3+h) - f(3) = 18h + 3h^2 = h(18+3h)$$
$$\text{slope at } x = 3 = \lim_{h\to 0}\frac{f(3+h) - f(3)}{h}$$
$$= \lim_{h\to 0}\frac{h(18+3h)}{h}$$
$$= \lim_{h\to 0}(18+3h)$$
$$= 18$$

Activity 6

This is one more interesting, although there is nothing new. Look at how we proceed:

$$\begin{aligned} f(x) &= x^2 \\ f(x+h) &= (x+h)^2 = x^2 + 2xh + h^2 \\ f(x+h) - f(x) &= x^2 + 2xh + h^2 - x^2 = 2xh + h^2 \\ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

Thus the slope or gradient of $f(x) = x^2$ at any point x is $2x$ (which we shall write as $f'(x) = 2x$).

Activity 7

There is nothing new in the question. You only have to substitute for $x + h$ very carefully.

$$\begin{aligned} f(x) &= 2x^2 - 5x + 6 \\ f(x+h) &= 2(x+h)^2 - 5(x+h) + 6 \\ &= 2(x^2 + 2xh + h^2) - 5x - 5h + 6 \\ &= 2x^2 + 4xh + 2h^2 - 5x - 5h + 6 \end{aligned}$$

It is important that $x + h$ is calculated correctly. If you make a mistake in the substitution for $f(x + h)$ you will not get the right answer.

$$f(x+h) - f(x) = 4xh + 2h^2 - 5h = (4x + 2h - 5)\,h$$

$$\begin{aligned} \text{gradient at any point } x &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{h(4x + 2h - 5)}{h} \\ &= \lim_{h \to 0} (4x + 2h - 5) \\ &= 4x - 5 \end{aligned}$$

Activity 8

$$\begin{aligned}
f(x) &= 5x^2 - 4x + 7 \\
f(x+h) &= 5(x+h)^2 - 4(x+h) + 7 \\
&= 5(x^2 + 2xh + h^2) - 4x - 4h + 7 \\
&= 5x^2 + 10xh + 5h^2 - 4x - 4h + 7 \\
f(x+h) - f(x) &= 5x^2 + 10xh + 5h^2 - 4x - 4h + 7 - (5x^2 - 4x + 7) \\
&= 10xh + 5h^2 - 4h \\
&= h(10x + 5h - 4) \\
\text{slope at any point } x &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{h(10x + 5h - 4)}{h} \\
&= \lim_{h\to 0} (10x + 5h - 4) \\
&= 10x - 4
\end{aligned}$$

The slope at $x = 1$ is $10(1) - 4 = 6$
i.e. simplify substitute $x = 1$

Activity 9

1. 3
2. 6
3. –5
4. –22

Activity 10

1. $6x$
2. $-6x$
3. $30x$
4. $-68x$

Activity 11

Be warned again: It is crucial to get the calculation of $f(x + h)$ correct.

$$
\begin{aligned}
f(x+h) &= a(x+h)^2 + \mathrm{b}(x+h) + \mathrm{c} \\
&= \mathrm{a}(x^2 + 2hx + h^2) + \mathrm{b}x + \mathrm{b}h + \mathrm{c} \\
&= ax^2 + 2ahx + ah^2 + \mathrm{b}x + \mathrm{b}h + \mathrm{c} \\
f(x+h) - f(x) &= 2ahx + ah^2 + \mathrm{b}h = h(2ax + ah + \mathrm{b}) \\
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{h(2ax+ah+b)}{h} \\
&= \lim_{h\to 0}(2ax + ah + b) \\
&= 2ax + b
\end{aligned}
$$

The derivative of $f(x) = ax^2 + \mathrm{b}x + \mathrm{c}$ is $2ax + b$
written $\mathrm{f}'(x) = 2ax + 6$

Activity 12

1. $6x + 4$
2. $-4x + 5$
3. $-8x - 7$
4. $10x - 13$

Activity 13

1.

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
g(x+h) &= (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3 \\
\therefore g(x+h) - g(x) &= 3x^2h + 3xh^2 + h^3 \\
g'(x) &= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0}(3x^2 + 3xh + h^2) \\
&= 3x^2
\end{aligned}
$$

2.

$$
\begin{aligned}
p(x+h) &= \frac{1}{x+h} \\
\therefore p(x+h) - p(x) &= \frac{1}{x+h} - \frac{1}{x} \\
&= \frac{x-(x+h)}{x(x+h)} \\
&= \frac{-h}{x(x+h)}
\end{aligned}
$$

$$\text{So, } p'(x) = \lim_{h\to 0}\frac{p(x+h)-p(x)}{h}$$

$$= \lim_{h\to 0}\frac{-h}{x(x+h)h} \qquad \left[\frac{-h}{x(x+h)} \div h = \frac{-h}{x(x+h)} \times \frac{1}{h}\right]$$

$$= \lim_{h\to 0}\frac{-1}{x(x+h)}$$

$$= \frac{-1}{x^2} \qquad \text{(since } x+h \to x \text{ as } h \to 0\text{)}$$

3. $c(x) = k$ for all values of x

$\therefore c(x+h) = k$

$\therefore c(x+h) - c(x) = k - k = 0$

$$\text{So, } c'(x) = \lim_{h\to 0}\frac{c(x+h)-c(x)}{h}$$

$$= \lim_{h\to 0}\frac{0}{h}$$

$$= \lim_{h\to 0} 0 \qquad \text{(NB. Simplify } \frac{0}{h} \text{ before taking the limit!)}$$

$$= 0$$

Activity 14

1. $f'(x) = 4.x^{4-1} = 4x^3$

2. $f'(x) = 5.x^{5-1} = 5x^4$

3. $f'(x) = -2.x^{-2-1} = -2x^{-3} (= \frac{-2}{x^3})$

4. $f'(x) = nx^{n-1}$

Activity 15

1. $f'(x) = 3x^2 - 2x^1 + 1.x^0 + 0 = 3x^2 - 2x + 1$

2. $g'(x) = 3(2x^1) - 4(1.x^0) + 0 = 6x - 4$

3. $h'(x) = -\frac{1}{2}(4x^3) + 7(2x^1) - 0 + 3(-2x^{-3}) - 10(-4x^{-5})$

$= -2x^3 + 14x - 6x^{-3} + 40x^{-5}$

(note that $\frac{3}{x^2} = 3x^{-2}$ and $\frac{10}{x^4} = 10x^{-4}$)

4. $i(x) = +x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{4}} \quad (\sqrt[n]{x} = x^{\frac{1}{n}})$

$\therefore \ i'(x) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{4}x^{-\frac{3}{4}}$

(note that the derivative of $x^{\frac{1}{2}} = \frac{1}{2}.x^{\frac{1}{2}-1}$, etc.)

5. $j(x) = (x^2+1)(x-3) = x^3 - 3x^2 + x - 3$

$\therefore j'(x) = 3x^2 - 6x + 1$

Lesson 3

Activity 1

1. Point: $(-1;3)$
   Gradient: $f'(x) = 2x$
   $\therefore$ at $x = -1 : f'(-1) = -2$
   Equation: $y - 3 = -2(x - (-1))$
   $y = -2x + 1$

2. Point: $(2; f(2))$ i.e. $(2;8)$ since $f(2) = 8$
   Gradient: $f'(x) = 3x^2$
   $\therefore$ at $x = 2 : f'(2) = 3(2)^2 = 12$
   Equation: $y - 8 = 12(x - 2)$
   $y = 12x - 16$

3. Point: $(0; f(0))$ i.e. $(0;4)$
   Gradient: $f'(x) = -2$ (i.e. does not depend on x)
   $\therefore$ at $x = 0 : f'(x) = -2$
   Equation: $y - 4 = -2(x - 0)$
   $y = -2x + 4$

Activity 2

Gradient $= f'(x) = 3x^2$

$\therefore 3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

The gradient for $y = x^3$ is 12 at $x = -2$ and at $x = 2$ i.e. at the points (–2; –8) and (2; 8).

Equations: Through $(-2;-8)$: $y+8=12(x+2)$

$$y=12x+16$$

Through $(2;8)$: $y-8=12(x-2)$

$$y=12x-16$$

The tangents are parallel to each other as they have the same gradient.

Activity 3

1.

Step 1. Find $f'(x)$

$f(x)=x^2 \therefore f'(x)=2x$

Step 2. Solve $f'(x)=0$ (if possible) for stationary points

$2x=0$ when $x=0$

Step 3. Find when $f'(x)<0$ and when $f'(x)>0$

The easiest way to do this is on a number line with x –values above the line and the corresponding sign of $f'(x)$ below.

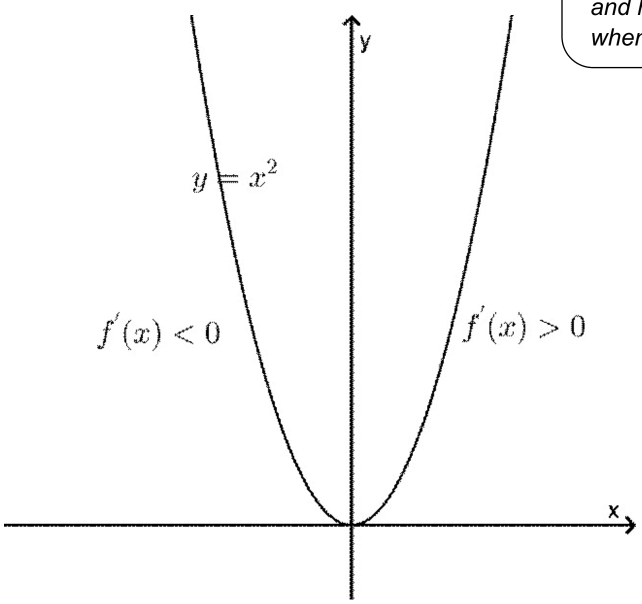
x	…–4	–3	–2	–1	0	1	2	3	4…
$f'(x)$	–	–	–	–	0	+	+	+	+

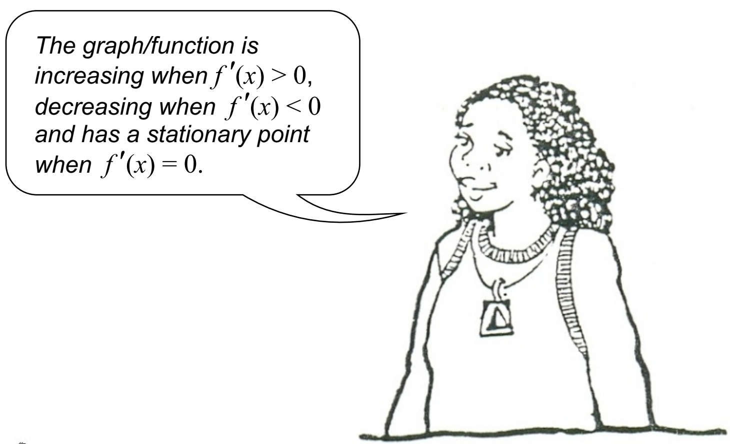
Thus $f(x)=x^2$ has a stationary point at $x=0$ i.e. at (0: 0)

is increasing when $x>0$

is decreasing when $x<0$

This is a confirmed by the graph $y=x^2$





2.

Step 1: $f(x) = 8 - 2x - x^2$ so $f'(x) = -2 - 2x$

Step 2: $f'(x) = 0$ (for stationary points)

$-2 - 2x = 0$

$x = -1$

i.e. stationary point at $(-1; f(-1)) = (-1; 9)$

Step 3:

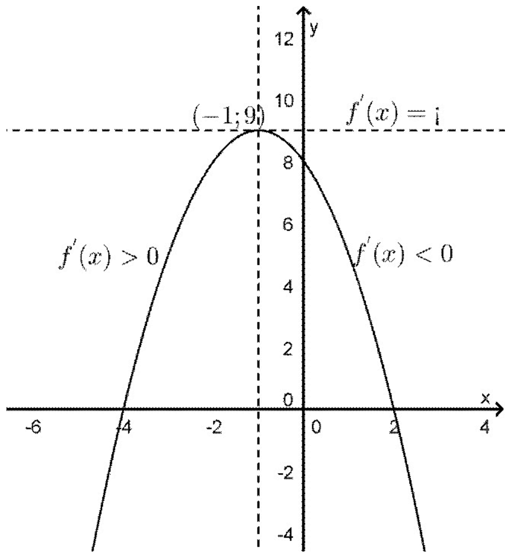
x	… −3	−2	−1	0	1	2	3 …
$f'(x)$	+	+	+	0	−	−	−
		/		—		\	

It is helpful to draw short lines under the table indicating increasing / and decreasing \.

f is increasing when $x < -1$ and decreasing when $x > -1$.

Note: At $x = -1$ (when $f'(x) = 0$) the graph stops increasing and starts decreasing. For this reason mathematicians usually say: f is increasing for $x \leq -1$, and f is decreasing for $x \geq -1$.

f is thought of as both increasing and decreasing at $x = -1$.



3.

Step 1: $f(x) = x^3 - 3x^2 - 9x$

$f'(x) = 3x^2 - 6x - 9$

Step 2: $f'(x) = 0$ (for stationary points)

$3x^2 - 6x - 9 = 0$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$x = -1$ or $x = 3$

Stationary points at $(-1; f(-1))$ and $(3; f(3))$
i.e. at $(-1;5)$ and $(3;-27)$

Step 3: $f'(x) = 3x^2 - 6x - 9$

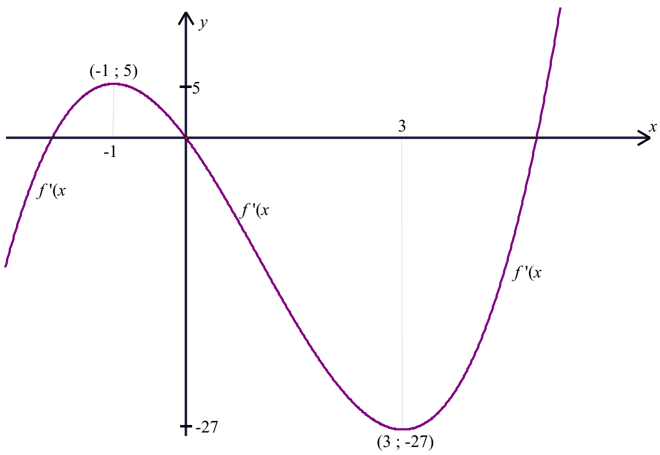
$= 3\,(x^2 - 2x - 3)$

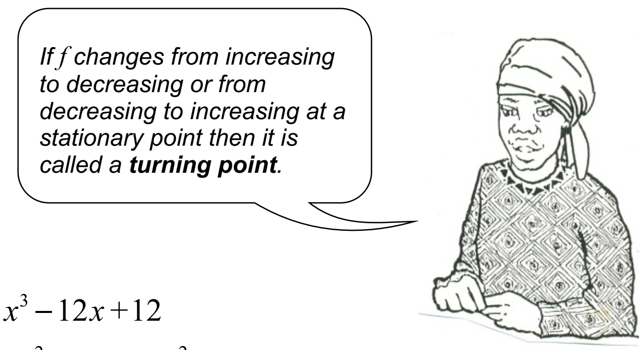
$= 3\,(x+1)(x-3)$

x	$\ldots -3$	-2	-1	0	1	2	3	4	5	$6 \ldots$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	0	$+$	$+$	$+$
		╱	—	╲	—	╱				

f is increasing when $x \le 1$ or $x \ge 3$
f is decreasing when $-1 \le x \le 3$

Here is a rough sketch of $y = f(x)$ (not to scale)





Activity 4

1.
Step 1: $f(x) = x^3 - 12x + 12$

$\therefore f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2)$

Step 2: $f'(x) = 0$ for stationary points

$\therefore 3x^2 - 12 = 0$

$x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x = 2$ or $x = -2$

Stationary points at $(-2; f(-2))$ and $(2; f(2))$
i.e. at $(-2;18)$ and $(2;-14)$

Step 3:

x	…	−5	−4	−3	−2	−1	0	1	2	3	4	5	…
$f'(x)$		+	+	+	0	−		−	0	+	+	+	
				↗	—		↘		—	↗			

At $x = -2$ f changes from increasing to decreasing so (–2; 18) is a maximum turning point.
At $x = 2$ f changes from decreasing to increasing so (2; –14) is a minimum turning point.

2.
Step 1: $g(x) = x^3$

$g'(x) = 3x^2$

Step 2: $g'(x) = 0$

$3x^2 = 0$

$x = 0$

$\therefore$ (0;0) is a stationary point

Step 3:

x				0			
$g'(x)$	+	+	+	0	+	+	+

Activity 5

1. a)

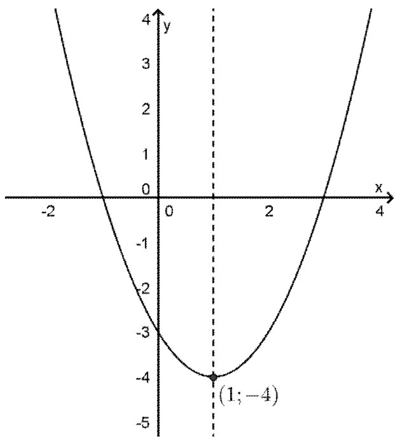
Step 1: $f(x) = x^2 - 2x - 3$

$f'(x) = 2x - 2 = 2(x-1)$

Step 2: $f'(x) = 0$ for stationary points

$2x - 2 = 0$

$x = 1$



Step3:

x				1			
$g'(x)$	−	−	−	0	+	+	+
			↘	—	↗		

f has a minimum turning point at $(1; f(1))$ i.e. at (1; –4)

Compare with using $x = -\frac{6}{2a} = \frac{-(-2)}{2(1)} = 1$

1. b)

Step 1: $g(x) = -x^2 + 4x + 12$

$g'(x) = -2x + 4 = -2(x - 2)$

Step2: $g'(x) = 0$ for stationary points

$-2x + 4 = 0$

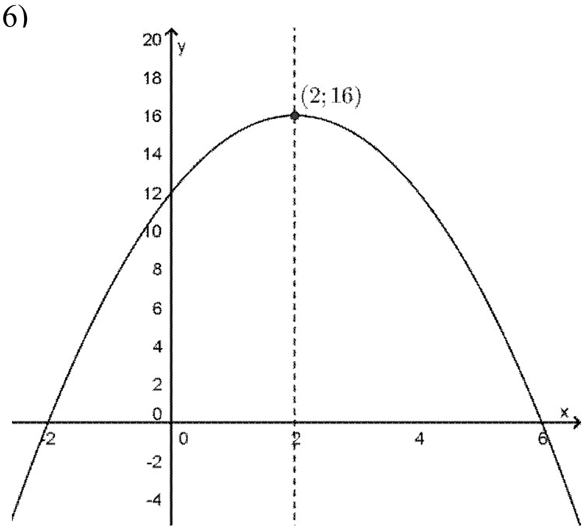
$x = 2$ (stationary point at (2;16))

Step 3:

x		2	
$g'(x)$	+ + +	0	− − −
	↗	—	↘

g has a maximum turning point at (2; 16)

Compare with $x = -\frac{b}{2a} = \frac{-4}{-2} = 2$



2.

Step 1: $p(x) = ax^2 + bx + c$

$\therefore p'(x) = 2ax + b \quad \left(= 2a\left(x + \frac{b}{2a}\right)\right)$

Step 2: $p'(x) = 0$ for stationary points

$2ax + b = 0$

$x = -\frac{b}{2a}$

Step 3: If $a > 0$:

x		$\frac{-6}{2a}$	
$p'(x)$	− − −	0	+ + +
	↘	—	↗

If $a < 0$:

x		$\frac{-6}{2a}$	
$p'(x)$	+ + +	0	− − −
	↗	—	↘

$\therefore \; p(x) = ax^2 + bx + c$ has

a minimum turning point at $x = -\dfrac{b}{2a}$ if $a > 0$

a maximum turning point at $x = -\dfrac{b}{2a}$ if $a < 0$

Lesson 4

Activity 1

1.
$$\begin{aligned} f(1) &= (1)^3 - 5(1)^2 + 7(1) - 2 = 1 \\ f(-1) &= (-1)^3 - 5(-1)^2 + 7(-1) - 2 = -15 \\ f(2) &= (2)^3 - 5(2)^2 + 7(2) - 2 = 0 \\ f(-2) &= (-2)^3 - 5(-2)^2 + 7(-2) - 2 = -44 \\ f(0) &= 0^3 - 5(0)^2 + 7(0) - 2 = -2 \end{aligned}$$

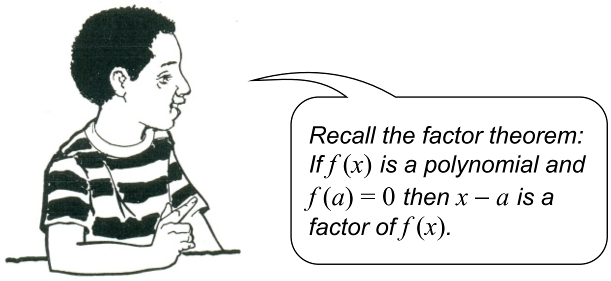
2.
$$\begin{array}{r|l} & x^2 - 3x + 1 \\ \hline x - 2 & x^3 - 5x^2 + 7x - 2 \\ & \underline{x^3 - 2x^2} \\ & \quad -3x^2 + 7x \\ & \quad \underline{-3x^2 + 6x} \\ & \qquad\qquad x - 2 \\ & \qquad\qquad \underline{x - 2} \\ & \qquad\qquad\quad 0 \end{array}$$

The quotient is $x^2 - 3x + 1$ and the remainder is 0. Notice also that $f(2) = 0$, which means that 2 is a root of $f(x) = 0$.

3. The fact that the remainder is 0 means that $x - 2$ is a factor of $f(x)$ and so we can write $f(x) = (x - 2)(x^2 - 3x + 1)$.

To solve $f(x) = 0$
we write $(x - 2)(x^2 - 3x + 1) = 0$

So $x - 2 = 0$
or $x^2 - 3x + 1 = 0$
$\therefore \; x = 2$
or $x = \dfrac{3 \pm \sqrt{9 - 4.1.1}}{2.1} = \dfrac{3 \pm \sqrt{5}}{2}$



The roots of $f(x) = 0$ are 2, $\dfrac{3 - \sqrt{5}}{2} \approx 0{,}4$ and $\dfrac{3 + \sqrt{5}}{2} \approx 2{,}6$.

We had to use the quadratic formula to solve $x^2 - 3x + 1 = 0$ as the only possible factors with integer coefficients would be $x + 1$ and $x - 1$, neither of which works.

(Remember, a quick check to see if $ax^2 + bx + c$ has factors with integer coefficients is to calculate $\Delta = b^2 - 4ac$. In this case $\Delta = 9 - 4 = 5 \approx$ perfect square so the quadratic formula is required.)

Notice also that $f(1) \approx 0$ and $f(-1) \approx 0$ which also tells us that –1 and 1 are not roots of $f(x) = 0$.

4. To find the stationary points, put $f'(x) = 0$.

$$f(x) = x^3 - 5x^2 + 7x - 2$$
$$f'(x) = 3x^2 - 10x + 7 \qquad (\Delta = 100 - 4.3.7 = 16!)$$
$$= (3x - 7)(x - 1)$$
$$\therefore f'(x) = 0 \text{ when } x = \frac{7}{3} \text{ or } x = 1$$

Stationary points are $(1; f(1))$ and $\left(\frac{7}{3}; f(\frac{7}{3})\right)$
i.e. (1; 1) and $(\frac{7}{3}; \frac{-5}{27})$.

5. We find where f is increasing or decreasing

x		1		$\frac{7}{3}$	
$f'(x)$	+ + +	0	− −	0	+ + +
	↗	—	↘	—	↗

Since f changes from increasing to decreasing at $x = 1$, (1;1) is a maximum turning point of $y = f(x)$.

Since f changes from decreasing to increasing at $x = \frac{7}{3}$, $(\frac{7}{3}; \frac{-5}{27})$ is a minimum turning point of $y = f(x)$.

Activity 2

1. $\frac{dy}{dx} = 9x^2 - 12x + 5$

2. $f(x) = x^2 + x^{-2}$, so $\frac{dy}{dx} = 2x - 2x^{-3}$

Activity 3

1. Step 1 x–intercepts: Put $y = 0$

$$x^3 - 9x = 0$$
$$x(x^2 - 9) = 0$$
$$x(x - 3)(x + 3) = 0$$
$$x = 0 \text{ or } x = +3 \text{ or } x = -3$$

$\therefore x$ –intercepts are (–3; 0); (0; 0); (3; 0)

y –intercept: Put $x = 0$

$$y = 0^3 - 9.0 = 0$$

$$\therefore\ y \text{ – intercept is } (0;0)$$

Step 2: Find $\frac{dy}{dx}$: $y = x^3 - 9x$

$$\frac{dy}{dx} = 3x^2 - 9 = 3(x^2 - 3)$$

Step 3: Stationary points

$$\frac{dy}{dx} = 0$$

$$3x^2 - 9 = 0$$

$$x^2 - 3 = 0$$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

i.e. $y = \left(-\sqrt{3}\right)^3 - 9\left(-\sqrt{3}\right) = -3\sqrt{3} + 9\sqrt{3} = 6\sqrt{3}$ and

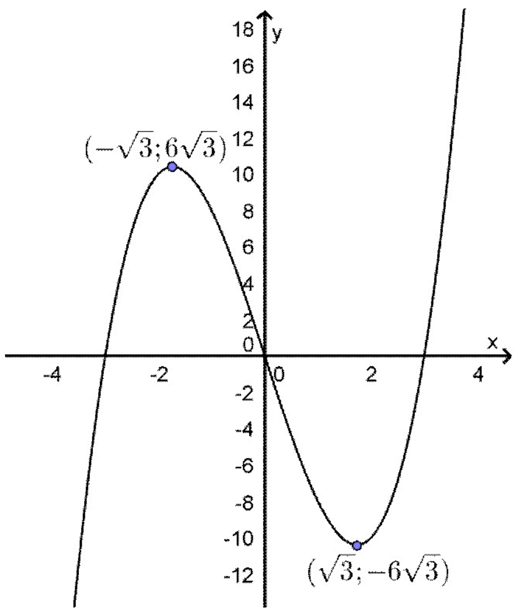
$y = \left(\sqrt{3}\right)^3 - 9\sqrt{3} = -6\sqrt{3}$

so stationary points at $(-\sqrt{3};+6\sqrt{3})$ and $(\sqrt{3};-6\sqrt{3})$

Or roughly, $(-1,7;10,4)$ and $(1,7;-10,4)$

Step 4: Table of signs (number line): $\frac{dy}{dx} = 3(x+\sqrt{3})(x-\sqrt{3})$

x		$-\sqrt{3}$		$\sqrt{3}$	
$\frac{dy}{dx}$	+ + +	0	– –	0	+ + +
	╱	—	╲	—	╱



The graph has a maximum turning point at $(-\sqrt{3};6\sqrt{3})$ and a minimum turning point at $(\sqrt{3;-6\sqrt{3})}$.

Step 5: Plot the points (approximately) and join them smoothly to obtain a sketch of the function.

2. Step 1:

x –intercepts $x^3 - 3x^2 + 2 = 0$

By inspection, 1 is a root of the equation (or, let $f(x) = x^3 - 3x^2 + 2$ then $f(1) = 1^3 - 3.1^2 + 2 = 0$)

By the factor theorem, $x - 1$ is a factor of $x^3 - 3x^2 + 2$.

$$
\begin{array}{r|l}
 & x^2 - 2x - 2 \\
x - 1 & x^3 - 3x^2 + 0x + 2 \\
 & \underline{x^3 - x^2} \\
 & \quad - 2x^2 + 0x \\
 & \quad \underline{- 2x^2 + 2x} \\
 & \qquad\quad - 2x + 2 \\
 & \qquad\quad \underline{- 2x + 2} \\
 & \qquad\qquad 0
\end{array}
$$

$\therefore y = (x-1)(x^2 - 2x - 2)$ $\quad (\Delta = 4 + 4.1.2 = 12 \neq p^2)$

To solve $x^2 - 2x - 2 = 0$ use the quadratic formula

$$x = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$\therefore$ intercepts: $(1;0),(1-\sqrt{3;0)}$ and $(1+\sqrt{3};0)$

y –intercept: $y = 0^3 - 3.0^2 + 2 = 2$
i.e. (0; 2)

Step 2: $\dfrac{dy}{dx} = 3x^2 - 6x = 3x(x-2)$

Step 3: Stationary points

$$\frac{dy}{dx} = 0$$

$$3x(x-2) = 0$$

$$x = 0 \quad \text{or} \quad x = 2$$

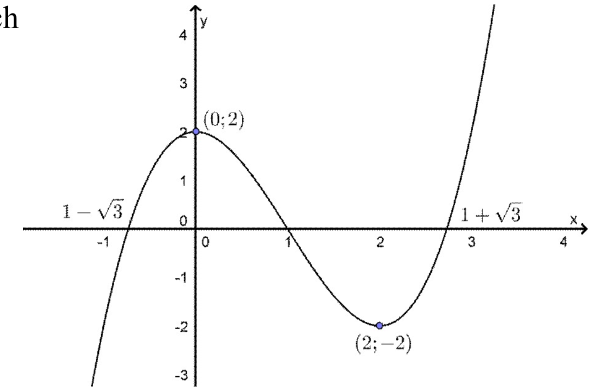
$\therefore$ Stationary points at (0; 2) and (2; –2)

Step 4: Table of signs (number line) $\dfrac{dy}{dx} = 3x(x-2)3x(x-2)$

x		0		2	
$\frac{dy}{dx}$	+	0	–	0	+
	╱	—	╲	—	╱

Maximum turning point at (0; 2)
Minimum turning point at (2; –2)

Step 5: Sketch



3. Step 1:

x –intercepts: $8 - x^3 = 0$

$x^3 = 8$

$x = 2$

$\therefore$(2; 0) is the only x –intercept
y –intercept: $y = 8 - 0^3 = 8$
i.e. (0; 8)

Step 2: $\dfrac{dy}{dx} = -3x^2$

Step 3: $\dfrac{dy}{dx} = 0$ when $-3x^2 = 0$

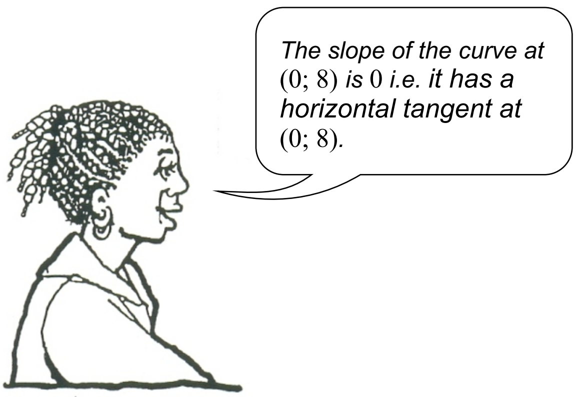
Only stationary point is (0; 8).

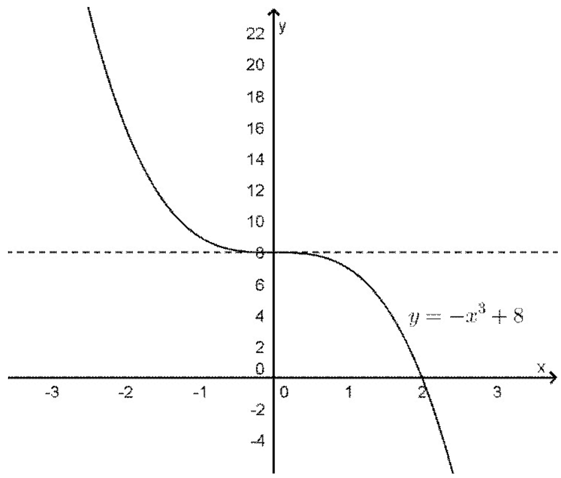
Step 4: $\dfrac{dy}{dx} = -3x^2$

x		0	
$\frac{dy}{dx}$	–	0	–
	↘	—	↘

The function decreases to $x = 0$, 'stops' for an instant at $x = 0$, then continues decreasing. (0; 8) is a point of inflection.

Step 5:





Activity 4

1. From the graph –1, 1 and 3 are zero of f (i.e. roots of the equation $f(x) = 0$).

Method 1: $f(-1) = (-1)^3 + a(-1)^2 + b(-1) + c = 0$

$$\therefore \quad a - b + c = 1 \qquad ①$$

$$f(1) = 1^3 + a(1)^2 + b(1) + c = 0$$

$$\therefore \quad a + b + c = -1 \qquad ②$$

$$f(3) = (3)^3 + a(3)^2 + b(3) + c = 0$$

$$\therefore \; 9a + 3b + c = -27 \qquad ③$$

We now solve these equations simultaneously:

② – ① : $2b = -2$

$\therefore b = -1$

③ – ② : $8a + 2b = -26$

$$8a + 2(-1) = -26$$

$$8a = -24$$

$$a = -3$$

Substitute $a = -3$ and $b = -1$ into ①

$$-3 - (-1) + c = 1$$

$$c = 3$$

Thus $a = -3$, $b = -1$ and $c = 3$

Method 2: Since –1, 1 and 3 are zeros of f

$(x + 1)$, $(x - 1)$ and $(x - 3)$ are factors of $f(x)$

$\therefore f(x) = p\,(x + 1)(x - 1)(x - 3)$ since a cubic function can only have 3 factors. For now, we used the constant p (although it is obvious $p = 1$ - but this may not always be the case).

$$\therefore p(x+1)(x-1)(x-3) \equiv x^3 + ax^2 + bx + c$$

$$p[x^3 - 3x^2 - x + 3] \equiv 1.x^3 + ax^2 + bx + c$$

$$px^3 - 3px^2 - px + 3p \equiv 1.x^3 + ax^2 + bx + c$$

Equating coefficients:

$p = 1,\; -3p = a,\; -p = b$ and $3p = c$

$\therefore \; a = -3,\; b = -1$ and $c = 3$

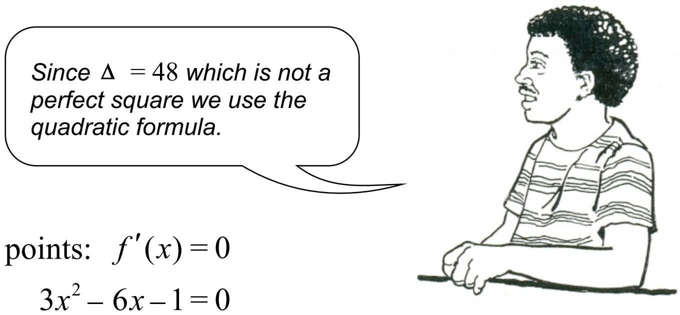
2. $y = x^3 - 3x^2 - x + 3 = f(x)$

y –intercept: put $x = 0, f(0) = 3$

$\therefore \; y$ –intercept is (0; 3)

so OK = 3 units

3. $f'(x) = 3x^2 - 6x - 1$ (Some notes may write $y' = 3x^2 - 6x - 1$)



At turning points: $f'(x) = 0$

$$3x^2 - 6x - 1 = 0$$

$$x = \frac{6 \pm \sqrt{48}}{6}$$

$$= \frac{6 \pm \sqrt{16.3}}{6} = \frac{6 \pm 4\sqrt{3}}{6} = 1 \pm \frac{2}{3}\sqrt{3}$$ (Simplest surd form)

4. Slope of $y = f(x)$ at $x = 2$ (and hence of the tangent)
$= f'(2) = 3(-2) - 6(2) - 1 = 1$
$\therefore\ m_{\tan} = 1$

At $x = 2,\ y = f(2) = 2^3 - 3.2^2 - 2 + 3 = -3$
so the coordinates of the point are (2; –3)

Equation of tangent: $y - (-3) = -1(x - 2)$

$$y = -x - 1$$

5. To find out whether the tangent cuts the curve again we need to solve $y = -x - 1$ and $y = x^3 - 3x^2 - x + 3$ simultaneously (not actually in the syllabus, but very easy!)

$$y = -x - 1 \qquad (1)$$

$$y = x^3 - 3x^2 - x + 3 \qquad (2)$$

Equating: $-x - 1 = x^3 - 3x^2 - x + 3 = (x-1)(x+1)(x-3)$

$$\therefore (x-1)(x+1)(x-3) + (x+1) = 0$$

$$(x+1)[(x-1)(x-3) + 1] = 0$$

$$(x+1)[x^2 - 4x + 4] = 0$$

$$(x+1)(x-2)^2 = 0$$

$$x = -1 \text{ or } x = 2$$

So the tangent cuts the graph again at $x = -1$ i.e. at the point (–1; 0) (one of the x –intercepts)

Lesson 5

Activity 1

1. $G(0) = 450$
   The sugar level without any injection of the insulin, i.e. without treatment with insulin, is 450 units.

2. $G(25) = -0{,}2\,(25)^2 + 450 = 325$
   The blood sugar level, an hour after the injection of 25 units of insulin, is 325.

3. $\frac{dG}{dx} = G'(x) = -0{,}4x$

   $G'(10) = -4$
   $G'(25) = -10$

   This is the rate at which the level of blood sugar is going down, after injection with 10 units and 25 units of insulin, respectively.

Activity 2

$\frac{dv}{dr}$ is the change of the volume of the vessels, with respect to a change in its radius. h, the length of blood vessel, is 80mm.

1. $\frac{dV}{dr} = 2\pi rh$

   $h = 80 \therefore \frac{dV}{dr} = 160\pi r$

2. a) when $r = 4mm$

   $\frac{dV}{dr} = 160\pi \cdot 4 = 640\pi\ mm^3/mm$

   b) when $r = 6mm$

   $\frac{dV}{dr} = 160\pi \cdot 6 = 960\pi\ mm^3/mm$

   c) when $r = 8mm$

   $\frac{dV}{dr} = 160\pi \cdot 8 = 1280\pi\ mm^3/mm$

Activity 3

1. Initial displacement is when $t = 0$ seconds.
   Therefore, $s(0) = 0$ metres.

2. At $t = 3$ seconds, $s\ (3) = 3\ (9) = 27$ m
   The particle is 27 metres away from the origin, i.e., from where it started.

3. Velocity = $s'(t)$
   $s'(t) = 6t$ m/s

4. $s'(3) = 3(6) = 18$
   After 3 seconds, the velocity is 18m/s.

Activity 4

This is a question you would normally have in physics or mechanics. You can answer it with the use of calculus.

1. We know that we can find velocity as the derivative of the displacement equation.

   Let velocity be $v(t)$.

   $$v(t) = s'(t) = 300 - 10t \text{ m/s}$$

   Initial velocity is when the journey of the shell starts, when $t = 0$

   $$v(0) = 300 - 10(0) = 300$$

   The initial velocity of the shell is 300 m/s.

2. The greatest height is reached when the shell turns to come down to the ground. This is a turning point. At this point velocity is zero. Then,

   $$\begin{aligned} 300 - 10t &= 0 \\ t &= 30 \text{ s} \end{aligned}$$

   The shell reaches its highest point after 30 seconds. The height then is

   $$\begin{aligned} s(30) &= 300(30) - 5(30)^2 \\ &= 9000 - 5(900) \\ &= 9000 - 4500 \\ &= 4500 \end{aligned}$$

   The greatest height reached by the shell is 4500 metres above the ground.

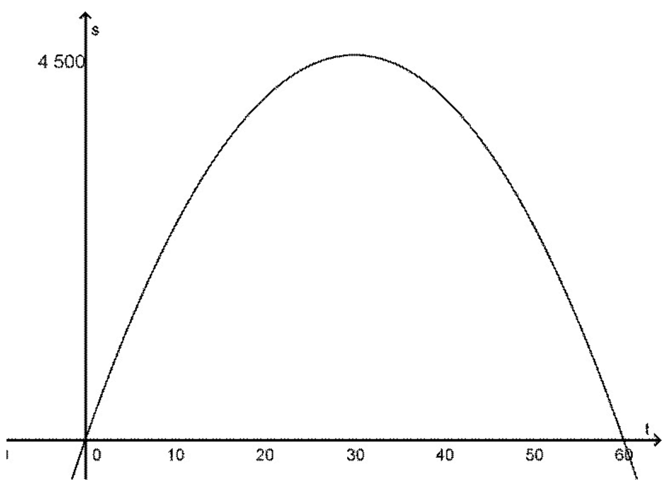
3. In physics we learn the following:
When a shell is fired from the ground, the time taken by the shell to reach its greatest height will be equal to the time it takes to reach the ground from the same height (assuming **no** friction from the air). Then, if the shell takes 30 seconds to reach its maximum height, it will take 30 seconds to fall to the ground from the same height. The total time for the shell to come back to the ground is then 60 seconds (1 minute).

We can show this mathematically: When the shell is on the ground its displacement (height) is 0.

$$s(t) = 0$$
$$300t - 5t^2 = 0$$
$$5t(60 - t) = 0$$
$$t = 0 \text{ or } t = 60$$

$t = 0s$ is when the shell is fired.
$t = 60s$ is when the shell lands back on the ground.

You can also sketch the graph that explains the motion of the shell. Look at this figure:



From the graph one can say what happened. The object moved 4 500 metres from the original position. It took 30 seconds to reach 4 500 metres. The object then turned and took 30 seconds to reach the original position.

Activity 5

Acceleration $a(t) = v'(t) = 2t - 1$

1. $a(3) = 2(3) - 1 = 5$
   After 3 seconds, acceleration is 5m/s^2.

2. $a(t) = 0$ when $2t - 1 = 0$
   $t = 0{,}5$ seconds.

3. When $t = 0$ (the start of the motion)
   $a(0) = 2(0) - 1 = -1$

   Acceleration is – 1 m/s^2 at the beginning. This means that the body is **decelerating** (or slowing down). This is sometimes called **retardation**.

Activity 6

Velocity, $v(t) = s'(t) = 12 - 4t$ m/s

after 3 seconds, $v(3) = 12 - 4(3) = 0$ m/s

When the velocity is zero, the object (body) is not moving. The body is said to be **stationary**.

Acceleration, $a(t) = v'(t) = -4$

At all times, acceleration is constant, equal to –4 m/s^2. This means the body is decelerating all the time it is in motion.

Activity 7

Let the two numbers be x and y.

We are told that $x + y = 20$

Therefore, $y = 20 - x$

The sum of the squares of the numbers is to be minimised.
Let the sum of the squares be $S(x)$.

$$S(x) = x^2 + y^2 = x^2 + x^2 - 40x + 400$$

$$S(x) = 2x^2 - 40x + 400 \quad \text{(Parabola with a minimum turning point)}$$

$$S'(x) = 4x - 40$$

At minimum, $S'(x) = 0$

Therefore, $4x - 40 = 0$

$$x = 10$$

$$y = 10$$

Activity 8

Let r be the radius and h the height, both in metres.

We are told that $h + r = 6$

Therefore, $h = 6 - r$

Let V be the volume of the cylinder in m^3

$$\therefore V = \pi r^2 h$$
$$= \pi r^2 (6 - r)$$
$$= 6\pi r^2 - \pi r^3$$

We need to find the maximum value of V. Therefore,

$$\frac{dV}{dr} = 12\pi r - 3\pi r^2$$

At the maximum, the derivative is equal to zero. Therefore,

$$12\pi r - 3\pi r^2 = 0$$
$$3\pi r(4 - r) = 0$$
$$r = 0 \quad \text{or} \quad r = 4$$

At $r = 4$, the volume is 0. This is obviously not the answer.

At $r = 4$, $h = 6 - 4 = 2$.

Therefore, maximum volume is

$$V = \pi r^2 h$$
$$= \pi .(4)^2 .2$$
$$= 32\pi \text{ m}^3$$
$$= 100,5 \text{ m}^3$$

Activity 9

1. We have to maximize the revenue.

$$R(x) = 6x - \frac{x^2}{1000}$$
$$R'(x) = 6x - \frac{x}{500}$$

$R'(x) = 0$ when $x = 3000$

To get maximum revenue she must produce 3000 items of school uniforms.

2. Profit is revenue less cost: $P(x) = R(x) - C(x)$

$$P(x) = 6x - \frac{x^2}{1000} - 2x$$

$$= 4x - \frac{x^2}{1000}$$

To get the highest profit, $P'(x) = 0$

$$P(x) = 4x - \frac{x^2}{1000}$$

$$P'(x) = 4x - \frac{x}{500}$$

$P'(x) = 0$ if $x = 2000$

At 3 000 units, she does not get the highest profit, although she gets the highest revenue. It is up to her to decide whether she wants to produce 2 000 units, which is maximum profit for her size factory, or 3000 units, and get more revenue but make less profit.

Feedback to Self-Check Exercises

Lesson 1

1. The functions are all continuous as the given points so we just substitute the x –values.

   a) $\lim_{x \to 0}(3x+2) = 3(0)+2 = 2$

   b) $\lim_{x \to 2}(x^2 - 2x + 6) = 2^2 - 2(2) + 6 = 6$

   c) $\lim_{x \to 4}(3x^2 - 6x - 8) = 3(4)^2 - 6(4) - 8 = 16$

   d) $\lim_{x \to 0} 5 = 5$ (since there is no x in the function it remains constant)

2. (It is a good idea to tabulate (make a table of values) your investigations.) The numerator remains positive as x approaches 2. The denominator changes. From the left, the denominator will be negative. The difference, $x - 2$, will be a very small negative number. The expression $\frac{x+2}{x-2}$ will be a very big negative number. Therefore

$$\lim_{x \to 2^-} \frac{x+2}{x-2} = -\infty$$

   From the right, the denominator will always be positive, therefore

$$\lim_{x \to 2^+} \frac{x+2}{x-2} = +\infty$$

3. Factorisation will again be an important part of the method for solving these problems.

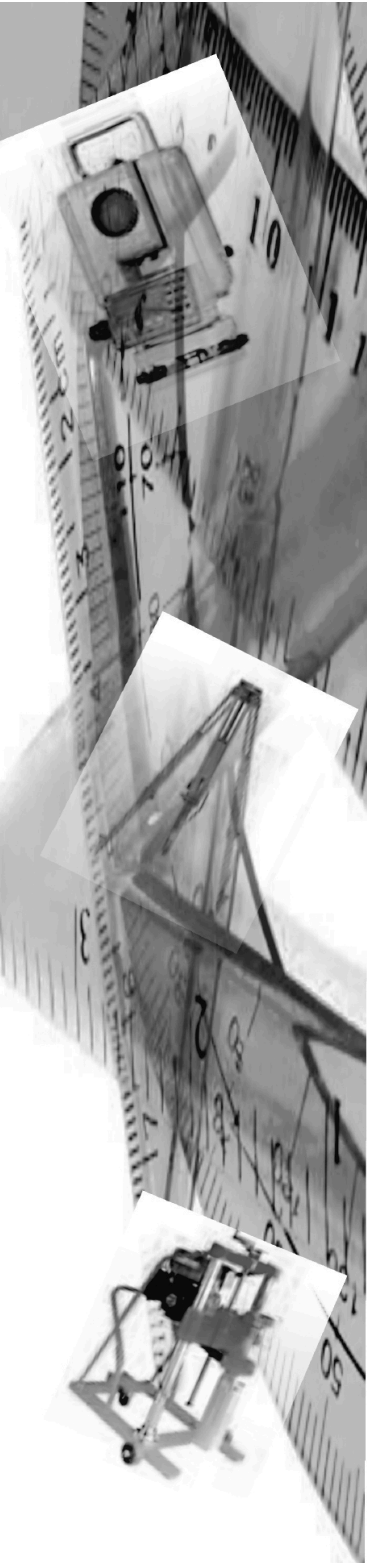
   a)
$$\begin{aligned}\lim_{x \to 2} \frac{x^2+x-6}{x-2} &= \lim_{x \to 2} \frac{(x+3)(x-2)}{x-2}\\ &= \lim_{x \to 2}(x+3)\\ &= 2+3\\ &= 5\end{aligned}$$

   b)
$$\begin{aligned}\lim_{x \to -2} \frac{3x^2+4x-4}{x+2} &= \lim_{x \to -2} \frac{(3x-2)(x+2)}{x+2}\\ &= \lim_{x \to -2}(3x-2)\\ &= 3(-2)-2\\ &= -8\end{aligned}$$

4. $f(x) = x^2 + x - 1$

   a) $f(2) = 2^2 + 2 - 1 = 5$

   b) $f(p) = p^2 + p - 1$



c) $f(1+h) = (1+h)^2 + (1+h) - 1$

$$= 1 + 2h + h^2 + 1 + h - 1$$

$$= h^2 + 3h + 1$$

d) $f(x+h) = (x+h)^2 + (x+h) - 1$

$$= x^2 + 2xh + h^2 + x + h - 1$$

Lesson 2

1. Average gradient $= \dfrac{\Delta y}{\Delta x}$

$$= \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{27 - 1}{2}$$

$$= 13$$

2. a) $p'(x) = \lim\limits_{h \to 0} \dfrac{p(x+h) - p(x)}{h}$

$$= \lim_{h \to 0} \frac{[3(x+h)^2 + 1] - [3x^2 + 1]}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h}$$

$$= \lim_{h \to 0} (6x + 3h)$$

$$= 6x$$

b) $q'(x) = \lim\limits_{h \to 0} \dfrac{q(x+h) - q(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$$

$$= \lim_{h \to 0} \frac{2x - 2(x+h)}{x(x+h)h}$$

$$= \lim_{h \to 0} \frac{-2h}{x(x+h)h}$$

$$= \lim_{h \to 0} \frac{-2}{x(x+h)}$$

$$= \frac{-2}{x^2}$$

3. a) 7   b) $2x$

c) $6x$ d) $-2x + 6$

e) $4x + 3$

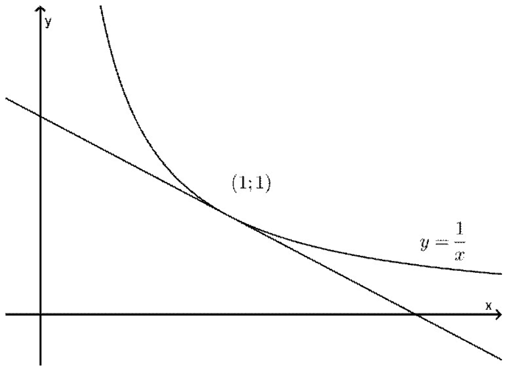
f) $f(x)=\dfrac{x^2-5x+6}{x^2-3x}=\dfrac{(x-3)(x-2)}{x(x-3)}=\dfrac{x-2}{x}=1-\dfrac{2}{x}$

$\therefore f'(x)=2x^{-2}=\dfrac{2}{x^2}$

Lesson 3

1. $h(x)=\dfrac{1}{x}=x^{-1}$

$\therefore h'(x)=-x^{-2}=\dfrac{-1}{x^2}$



So at (1; 1) gradient of $y=h(x)$ is $h^1(1)=-1$
$\therefore$ gradient of the tangent to $y=h(x)$ at (1; 1) is $m=-1$.

Equation is: $y-1=-1(x-1)$

$y=-x+2$

2. Step 1: $f(x)=2x^3-3x^2-36x+10$

$\therefore f'(x)=6x^2-6x-36$

$=6(x^2-x-6)$

$=6(x-3)(x+2)$

Step 2: For stationary points $f'(x)=0$

$6(x-3)(x+2)=0$

$x=-2$ or $x=3$

Step3:

x		-2		3	
$f'(x)$	+	0	−	0	+
	╱	—	╲	—	╱

The function has a maximum turning point at (–2; 54)
and a minimum turning point at (3; –71).

3. Let $f(x)=x^2-2x+5$

Step 1: $f^1(x)=2x-2=2(x-1)$

Step 2: Stationary point at $f'(x)=0$

$2(x-1)=0$

$x=1$

Minimum turning point at (1;4) (since $a>0$)
Gradient of the parabola, and hence also of the tangent, at the turning point is $m=0$.
$\therefore$ equation of a tangent through (1; 4) is

$y-4=0(x-1)$

$y=4$ (horizontal line!)

Lesson 4

1. a) This question could obviously be solved using what you have learnt about quadratic functions, but we are going to solve it using calculus.

$$y = f(x) = 2x^2 - 8x + 6$$

Step 1: Intercepts:

y –intercept: $y = f(0) = 6$ $(0;6)$ is the y –intercept

x –intercepts: $y = 0$

$$2x^2 - 8x + 6 = 0$$
$$x^2 - 4x + 3 = 0$$
$$(x-1)(x-3) = 0$$
$$\therefore x = 1 \quad \text{or} \quad x = 3$$

(1; 0) and (3; 0) are the x –intercepts.

Step 2: $\frac{dy}{dx} = 4x - 8 = 4(x-2)$

Step 3: Stationary points: $\frac{dy}{dx} = 0$

$$4(x-2) = 0$$
$$x = 2$$

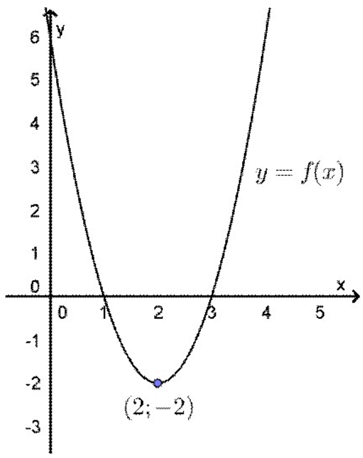
$f(2) = -2$ $\therefore$(2; –2) is the only stationary point.

Step 4: Table of signs (number line) $\frac{dy}{dx} = 4(x-2)$

x		2	
$\frac{dy}{dx}$	– ↘	0	+ ↗

$\therefore$ (2; –2) is a minimum turning point

Step 5: a)



b) $y = g(x) = x^3 - 3x^2 + 4$

Intercepts:

y –intercept: $y = g(x) = 4$ $\therefore (0;4)$ is the y –intercept

x –intercept: $y = g(x) = 0$

(possible factors: $x \pm 1, x \pm 2, x \pm 4$)

$g(-1) = 0$ $\therefore x + 1$ is a factor of $g(x)$.

By long division (or otherwise)

$$y=(x+1)(x^2-4x+4)=(x+1)(x-2)^2$$

$\therefore\ y=0$ when $x=-1$ or $x=2$ (double root, which indicates that the x –axis is a tangent to $y=g(x)$ at $x=2$.

$\therefore\ x$ –intercepts are (–1; 0) and (2; 0)

Stationary points: $\frac{dy}{dx}=g^1(x)=3x^2-6x=3x(x-2)$

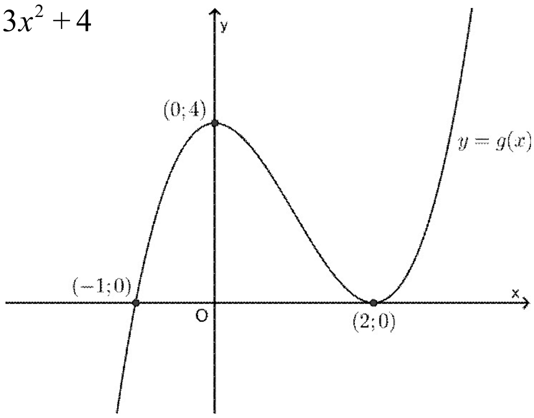
$\frac{dy}{dx}=0$ when $x=0$ or $x=2$

$\therefore$ (0; 4) and (2; 0) are stationary points

x		0		2	
$g'(x)$	+	0	–	0	+
	╱	—	╲	—	╱

(0; 4) is a maximum turning point and (2; 0) a minimum turning point.

Sketch of $y=x^3-3x^2+4$



c) $y=h(x)=-x^3+12x=x(12-x^2)$

Intercepts:
y –intercept: (0; 0)
x –intercept: $x(12-x^2)=0$
$\therefore x=0$ or $x^2=12$

$$x=\pm\sqrt{12}=\pm\sqrt{4.3}=\pm2\sqrt{3}$$

$\therefore\ x$ –intercepts $(0;0)$; $(-2\sqrt{3};0)$; $(2\sqrt{3};0)$; $(2\sqrt{3}\approx3,5)$

Stationary points: $\frac{dy}{dx}=h'(x)=-3x^2+12$

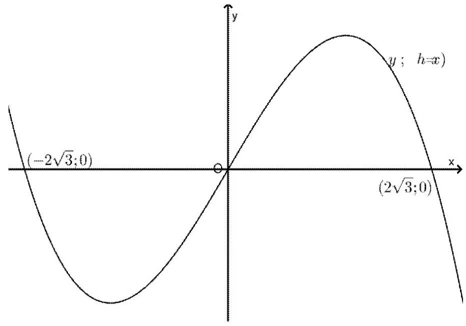
$$=-3(x^2-4)$$

$$=-3(x-2)(x+2)$$

$\therefore$ Stationary points (–2; –16) and (2; 16)

x		–2		2	
$h'(x)$	–	0	+	0	–
	╲	—	╱	—	╲

(–2; –16) is a minimum turning point.
(2; 16) is a maximum turning point.



d) $y = i(x) = -x^3 - 3x^2 - 3x + 7$

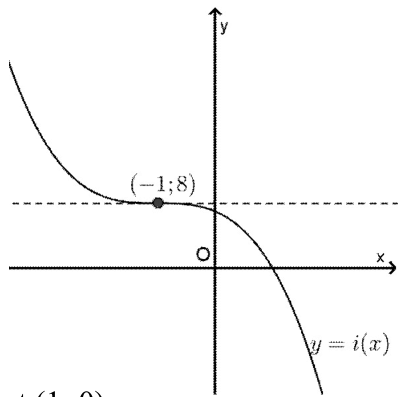
Intercepts:
y –intercept: (0; 7)
x –intercepts: $y = 0$ (possible factors $x \pm 1$, $x \pm 7$)

$i(1) = 0$ $x - 1$ is a factor of $i(x)$
$i(x) = -(x - 1)(x^2 + 4x + 7)$
$x^2 + 4x + 7 = 0$ has no root (Since $\Delta = 16 - 28 < 0$)
$\therefore$ only x –intercept is (1; 0)

Stationary points:
$$\begin{aligned} y' &= -3x^2 - 6x - 3 \\ &= -3(x^2 + 2x + 1) \\ &= -3(x+1)^2 \end{aligned}$$
$\therefore$ $y' = 0$ when $x = -1$ so only stationary point is (–1; 8)

x		-1	
y'	$-$	0	$-$
	↘	—	↘

(–1; 8) is a point of inflection.



2. a) From the sketch B is the point (1; 0)

b) (0; 0) is the y –intercept so $d = 0$

c) B(1; 0) lies on the curve so when $x = 1$, $y = 0$
$$\therefore 0 = -1 + b + c$$
$$b + c = 1 \quad ①$$
B(1; 0) is a turning point $\therefore$ when $x = 1$, $y' = 0$
$$y' = -3x^2 + 2bx + c$$
$$0 = -3 + 2b + c$$
$$2b + c = 3 \quad ②$$
② – ① gives $b = 2$ and hence c = –1.

d) $y = -x^3 + 2x^2 - x$

$\therefore y' = -3x^2 + 4x - 1$

$= -(3x^2 - 4x + 1)$

$= -(3x - 1)(x - 1)$

$\therefore \ x = \frac{1}{3}$ (at A) or $x = 1$ (at B)

At A, $y = -(\frac{1}{3})^3 + 2(\frac{1}{3})^2 - (\frac{1}{3}) = \frac{-4}{27}$

$\therefore$ A is the point $(\frac{1}{3}; \frac{-4}{27})$

Lesson 5

1. a) The body is momentarily at rest when $v(t) = 0$.

Therefore, $4t - t^2 = 0$

$t(4 - t) = 0$

$t = 0$ or 4

The body is momentarily at rest when time is 4 seconds.

Note: $t = 0$ is a trivial answer.

b) Acceleration, $a(t) = v'(t)$

$v'(t) = 4 - 2t$

Acceleration when the body is momentarily at rest is a (4).

Trivial case: $a(0) = 4$

The body starts moving with an acceleration of 4 m/s^2.

$a(4) = -4$

The body is momentarily at rest when it is decelerating at –4 m/s^2.

2. Let the length of the pen be x.
Let the width of the pen be y.

Perimeter: $2(x+y)=600m$

Therefore, $y=300-x$

Area of pen $xy=x(300-x)$

$A(x)=300x-x^2$

To maximise area, $A'(x)=0$

$A'(x)=300-2x$

$300-2x=0$

$x=150$

Therefore, $y=150$

Maximum area $=(150\times 150)\ \text{m}^2$

$=22500\ \text{m}^2$

You may have noticed already that the rectangle with the biggest area is a square.

3. $T(x)=-x^2+50x+150$ is the number of tourists.
x is rainfall in cm.

a) The highest number of tourists is the maximum of $T(x)$.

$T'(x)=-2x+50$

At maximum, $T'(x)=0$

$-2x+50=0$

$x=25$

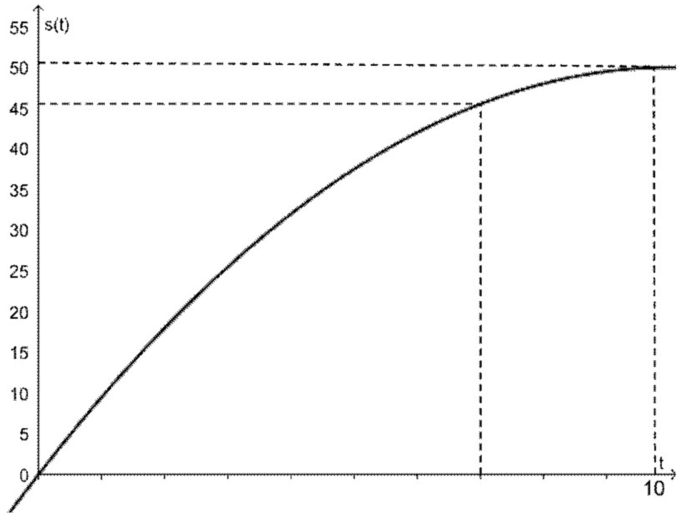
The rainfall that will produce the highest number of tourists is 25cm.

b) Maximum number of tourists $=T(25)$

$=-625+1250+150$

$=775$

4. a)



b) $x(5) = 10(5) - 0,5(25) = 37,5$
After 5 seconds, the crate is 37,5 metres down the slide.

$x(7) = 10(7) - 0,5(49) = 45,5$
After 7 seconds, the crate is 45,5 metres down the slide.

c) The slide is 50m long.

$$50 = 10t - 0,5t^2$$
$$10t - 0,5t^2 - 50 = 0$$
$$t^2 - 20t + 100 = 0$$
$$(t - 10)^2 = 0$$
$$t = 10$$

A crate will take 10 seconds to go down the whole slide. According to the problem, the crate will just stop at 50 metres. This means that the packers get the crate just when it stops.

Is it possible that the engineer did not want the crate to go over and break if a worker failed to pick it up on time? Maybe.

d) The crates start at a very high speed. The slope of the function $x'(t)$ is negative. This means that the speed is decreasing as the crate moves down the slide. By the time it reaches the point where it is about to be packed, its speed will not be dangerous for those who are packing. These are important safety measures at work. What safety measures do you have at your workplace?

$$\begin{aligned}\text{Velocity} &= x'(t)\\ &= 10 - t\end{aligned}$$

after 1 second, velocity $= 10 - 1 = 9$ m/s.
after 8 seconds, velocity $= 10 - 8 = 2$ m/s.