Topic 1: Number and Number Relationships

NASCA Mathematics

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# Sub-Topic 1: Exponents, Surds and Logarithms

I like to make things in my garage out of wood. I learnt most of what I know from my father. But growing up, I often got really frustrated when he insisted that I spend time learning how to just screw in screws straight or cut simple miter joints or how to hand plane. I wanted to make swords and tables and a wine rack for my mum. As much as I protested, he always said “You first need to learn the basics and how and when to use your basic tools.”

This sub-topic on exponents, surds and logarithms is kind of like learning how to use a screwdriver, miter box and hand plane. On its own, it does not seem very useful or important but it really is because the basic skills you learn now, will help you do some really cool things later on like solve quadratic equations to figure out how far a ball will travel or work out how much you have to pay every month for a R1 million loan.

## Unit 1: The Laws of Exponents

#### Learning Outcomes

By the end of this unit, you should be able to:

1. State and use the basic laws of exponents;
2. Extend the basic laws of exponents from integer exponents to rational exponents (exponents that are fractions); and
3. Use the laws of exponents to simplify expressions containing rational exponents.

#### Introduction

Mathematicians are lazy! I mean this in a good way. Mathematicians are always trying to find ways to make things simpler, easier and faster. You probably know that *multiplication* is just a shorthand method (a quicker way of saying the same thing) for *repeated addition*. Think about this.

Which expression would you rather write?

or

1. Which expression is easier and quicker to say?
2. Which expression is easier and quicker to read?
3. Which expression is likely to cause the fewest mistakes?

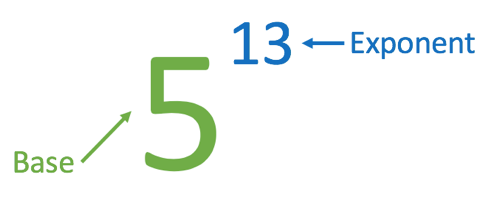
I hope you agree that is the better option. Using multiplication instead of repeated addition becomes even more important when we are working with a mixture of constants (numbers we know the value of) and variables (letters that stand in place of numbers which we might not know the value of). Think about the differences between writing, saying, reading and calculating with each of the following expressions.

and

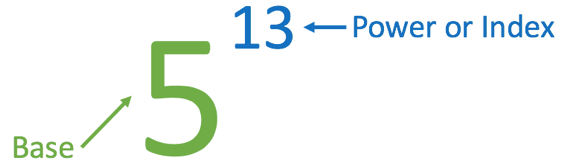
But what about repeated multiplication. Like multiplication is a shorthand for repeated addition, exponents give us a shorthand for repeated multiplication. Which expression would you rather use?

or

Let’s unpack a bit. The image below tells which part is the **base** (the number being multiplied) and which part is the **exponent** (the number of times the base is being multiplied).



The term “exponent” is sometimes also called “**power**” or “**index**”. **Exponent**, **power** and **index** all mean the same thing.



### Activity 1: Calculating with Simple Exponents

|  |  |
| --- | --- |
| Bullseye | Purpose In this activity, you are going to get some practice writing expressions with exponents and finding the value of expressions written with exponents. You will also get some practice calculating expressions with exponents with a calculator. |
| Stopwatch | Suggested Time You will need about 10 minutes. |
| Pencil | What You Will Need  * A pen * Some blank paper or a notebook * A calculator |

#### Tasks

1. Write the following expressions in a simpler way:
2. Write the following expressions without exponents (in expanded form) if possible.
3. Without using your calculator, work out what the values of these expressions are:
4. Use your calculator to work out what the values of these expressions are:

#### Guided Reflection

1. We had to write each expression in a simpler way.
   1. This was repeated multiplication, so we need to use exponents.
   2. Be careful here. While there is repeated multiplication, it is not always the same base being multiplied by itself.
   3. This was repeated addition not multiplication, so no exponents are needed.
2. Here we needed to write each expression without exponents.
3. No calculators were allowed here.
   1. (You need to multiply all the numerators together and then multiply all the denominators together)
4. In this question, you could use your calculator. Most of the time, when you are working with exponents, you will use the exponent function on your calculator. Different calculators work slightly differently but the two main types out there are Sharp and Casio.

If you don’t know how to use the exponent function on your calculator, you should watch [this video](https://www.youtube.com/watch?v=H0BhadVj8JE) if you have a Sharp calculator and [this video](https://www.youtube.com/watch?v=B6GgcF6uz_k) if you have a Casio calculator.

You say as “thirteen to the fifth power” or just “thirteen to the five”.

You say as “nine to the seventh power” or just “nine to the seven”

* 1. (here you needed to either work out what one divided by five was first, or use the bracket function on your calculator)

This might seem like a strange answer. It is less than one but not negative. We will see why we get this answer in a little while.

OK, now that we have a basic grasp of what exponents are and how they work, lets go a little into the exponent pool and see if we can figure out some rules for exponents.

### Activity 2: Exponent Law and Order

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the basic exponent laws. |
| Stopwatch | Suggested Time You will need about 25 minutes. |
| Pencil | What You Will Need  * A pen * Some blank paper or a notebook |

#### Tasks

1. Consider the expression .
2. Expand this expression fully.
3. Now write your expanded expression with exponents again.
4. What do you notice about the exponent in the expression we have now and the exponents in the expression we started with? Write a general rule for .
5. Consider the expression .
6. Expand this expression fully.
7. Now write your expanded expression with exponents again.
8. What do you notice about the exponent in the expression we have now and the exponents in the expression we started with? Write a general rule for .
9. What do you think is equal to?
10. Consider the expression .
11. Expand this expression fully.
12. Now write your expanded expression with exponents again.
13. What do you notice about the exponent in the expression we have now and the exponents in the expression we started with? Write a general rule for .
14. Consider the expression .
15. Expand this expression fully.
16. Now write your expanded expression with exponents again but this time with exponents on each base.
17. What do you notice about the exponent in the expression we have now and the exponents in the expression we started with? Write a general rule for
18. Consider the expression .
19. Expand this expression fully.
20. Now write your expanded expression with exponents again but this time with exponents on each base.
21. What do you notice about the exponent in the expression we have now and the exponents in the expression we started with? Write a general rule for

#### Guided Reflection

1. We are given the expression .
2. If we expand this expression, we get

.

1. Writing it with exponents again, we get .
2. We can see that the exponent in our second expression is just the *sum* of the exponents in our original expression. Therefore, in general we can say that .
3. We are given the expression .
4. If we expand this expression, we get
5. Writing it with exponents again, we get .
6. We can see that the exponent in our second expression is just the *difference* between the exponents in our original expression. Therefore, in general we can say that .
7. We can think of as where . Let’s say that then . But we also know that . This means that if then . This means that .

This is true no matter what the base is except when the base is zero. is undefined.

1. We are given the expression .
2. If we expand this expression, we get
3. Writing it with exponents again, we get .
4. We can see that the exponent in our second expression is just the *product* of the exponents in our original expression. Therefore, in general we can say that .
5. We are given the expression .
6. If we expand this expression, we get
7. Writing it with exponents again, we get .
8. We can see that the exponent in our second expression is just the *product* of the exponents in our original expression. Therefore, in general we can say that .
9. We are given the expression .
10. If we expand this expression, we get
11. Writing it with exponents again, we get .
12. We can see that the exponent in our second expression is just the *product* of the exponents in our original expression. Therefore, in general we can say that .

Well congratulations! You have just found most the basic laws of exponents. Here is a summary of what we know so far.

* times.

However, we have still not figured out what really means. Now we have the tools to crack this puzzle.

### Activity 3: Negative Exponents

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to negative exponents and give you some experience in working with negative exponents and manipulating expressions with negative exponents in them. |
| Stopwatch | Suggested Time You will need about 15 minutes. |
| Pencil | What You Will Need  * A pen * Some blank paper or a notebook |

#### Tasks

1. Consider the expression .
2. Expand this expression fully.
3. Use the relevant exponent law to simplify this exponent.
4. Now, what do you think is equal to? Write a general rule for .
5. Write each of the following expressions so that all exponents are positive.

#### Guided Reflection

1. We are given the expression .
2. If we expand this expression, we get
3. We know that . Therefore
4. In general, we can say that .
5. We need to write all the expressions with positive exponents.
6. We know that, in general, . This means that .
7. .

It does not matter if the base is a constant or variable. The same rule applies.

It does not matter if the exponent is a constant or variable. The same rule applies.

Be careful here. The exponent only applies to the base . When we take the base from the numerator to the denominator to make the exponent positive, the needs to stay in the numerator.

Here, we made it explicit that the exponent applied to the base . Remember that and this is why the stays in the numerator.

If moving a negative exponent from the numerator to the denominator makes it positive, then moving a negative exponent from the denominator to the numerator also makes it positive. In the solution, we can see why this must be the case.

Remember, the exponent is only applied to the base . Therefore, the variable must stay in the denominator.

Here the exponent is applied to the base . It is always best to leave the brackets in the expression to make sure everyone understands that the exponent applies to and not just .

Obviously, if then . Whenever we move a base and its exponent across the division sign from the numerator to the denominator or form the denominator to the numerator, the sign of the exponent changes.

Here is what we know so far.

* times.
* then

Let’s practice using these laws to simplify some expressions with exponents in them.

### Activity 4: Making Things Simpler

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the basic exponent laws. |
| Stopwatch | Suggested Time You will need about 60 minutes. |
| Pencil | What You Will Need  * A pen * Some blank paper or a notebook |

#### Tasks

1. Simplify each of these expressions as far as possible.
2. Simplify each of these expressions as far as possible.
3. Simplify each of these expressions as far as possible.

#### Guided Reflection

1. We were asked to simplify each expression as far as possible.
2. . We just need to apply the exponent law that says that . Therefore, .
3. . We cannot use the straight away because the bases are not the same. But we can make into something with base 2.

(recognise that )

(use the law that says )

(use the law that says )

1. . Here the bases ae not the same and there is no way to get them the same. The only possible thing we could do to simplify this expression any further would be to evaluate it with a calculator.
2. . The bases are already the same.

If you are confused about why then take a look at Topic 2 Sub-topic 1 Unit 1.

(use the law that says that )

1. . Remember that, if there are no brackets, the exponent only applies to the base right next to it.

(remember that the exponent outside the brackets applies to everything inside the brackets)

(here we recognised that )

(collect all the exponents on the sane bases and simplify)

(always right your final answers with positive exponents)

(if you are confused about why then take a look at Topic 2 Sub-topic 1 Unit 1)

(if you are confused about why then take a look at Topic 2 Sub-topic 1 Unit 1)

If you are confused about why then take a look at Topic 2 Sub-topic 1 Unit 1.

1. Again, we need to simplify each expression as much as possible.

(recognise that and that )

If any of the simplification of the exponents is confusing, then take a look at Topic 2 Sub-topic 1 Unit 1.

(we know that so we took the negative sign out and placed it in front of the )

## Unit 2: (Ab)surd Rational Exponents

#### Learning Outcomes

By the end of this unit, you should be able to:

#### Introduction

So far, every time we have seen an exponent it has been an integer, a whole number which is either positive or negative. But do exponents always have to be nice whole numbers? Can they be fractions maybe, like . But what does mean anyway?

### Activity 1: Fraction Exponents

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the basic exponent laws involving exponents that are fractions (also called rational exponents). |
| Stopwatch | Suggested Time You will need about 15 minutes. |
| Pencil | What You Will Need  * A pen * Some blank paper or a notebook |

#### Tasks

1. We know that .
2. If so, what does equal?
3. Therefore, what do you think means? Hint: What number multiplied by itself is equal to 4.
4. We know that .
5. If so, what does equal?
6. Therefore, what do you think means? Hint: What number multiplied by itself three times is equal to 27.
7. We know that .
8. If so, what does equal?
9. Therefore, what do you think means?
10. We know that . If so, what does equal?
11. We know that . If so, what does equal?

#### Guided Reflection

1. We already know that
2. So,
3. is the number that, when multiplied by itself is equal to . In other words, . But . This must mean that .
4. We already know that
5. So,
6. 3 is the number that, when multiplied by itself three times is equal to . In other words, or . But . This must mean that .
7. We already know that
8. So,
9. 2 is the number that, when multiplied by itself four times is equal to . In other words, or . But . This must mean that .
10. We already know that . This mean that
11. We already know that . This mean that

We have just discovered another few laws of exponents that deal with exponents that are not whole numbers. We saw that

* or
* or

Let’s think a bit more about what this all means.

We saw that . We say that is the **square root** of . This means that 2 is the number that must be multiplied by itself *twice* to equal 4.

We saw that . We say that is the **cube root** of . This means that 3 is the number that must be multiplied by itself *three times* to equal 27.

We saw that . We say that is the **fourth root** of . This means that 2 is the number that must be multiplied by itself *four times* to equal 27.

In general, we can say that . We say that is the **nth root** of . This means that is the number that must be multiplied by itself  *times* to equal .

is called a **root** or **radical**. Each part of it has its own special name.

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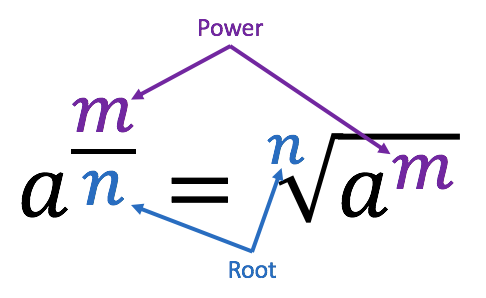
**NOTE**: Because we write the square root so often, we tend to leave out the degree. Whenever you see you can assume this means (the square root of ).

We also saw that the number at the top of the fractional exponent does not have to be 1.

could also be written as .

could also be written as .

This means that the number at the top of the fraction is the power and the number at the bottom of the fraction is the root.



Now spend a few minutes watching these great summary videos of all the basic exponent laws.

1. [Laws of indices Part 1 – Multiply, Divide, Power to a Power, Power of 0](https://www.fuseschool.org/topics/179#contents/1441) (05:30)

(<https://www.fuseschool.org/topics/179#contents/1441>)

1. [Laws of Indices Part 2: Negatives and Fractions](https://www.fuseschool.org/topics/179#contents/1586) (04:05)

(<https://www.fuseschool.org/topics/179#contents/1586>)

### Activity 2: Playing with Fraction Exponents

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will give you some practice in working with rational exponents (exponents that are fractions). |
| Stopwatch | Suggested Time You will need about 15 minutes. |
| Pencil | What You Will Need  * A pen * Some blank paper or a notebook |

#### Tasks

1. Simplify the following, writing the final answer with positive exponents.

#### Guided Reflection

1. We needed to simplify expression with rational exponents.
2. . All we really need to do here is apply our existing law which says that .

(lucky for us, the lowest common denominator of the fractions is simply 4)

1. . Nice easy one!
2. . When the question has numbers in it, we need to try and evaluate any roots.

(we know that an exponent of means the cube root.

(as you do more of these, you will easily recognise some of the most common perfect squares, cubes and fourth roots)

You could also answer this question this way.

(find the cube root of the numerator and the cube root of the denominator)

(you need to recognise that )

This is the best answer we have because there is no real number that we can multiply by itself that will equal -9. and . We would need to extend our knowledge of numbers to included non-real numbers to be able to go any further.

While the answer above is technically correct, we normally like to leave our answers even simpler. In ths case, we would just apply our rational exponent law.

1. . Even though this question might look scary, just apply the exponent rules one by one.

(remember to apply the exponent outside the bracket to every term inside the bracket)

Can we write in a better way or is this the best? We will see in a moment.

(We notice that we are dealing with the fourth root of 16 which we know is 2, so we can simplify further)

Now we can recognise that .

Now . We will look at more examples of this in the next activity.

You could also write this as

Watch the video called [Fractional exponent expressions 2](https://www.youtube.com/watch?v=4F6cFLnAAFc) (07:30) for video worked solution of this question.

(<https://www.youtube.com/watch?v=4F6cFLnAAFc>)

We cannot deal with the exponent of outside the bracket because we have more than one term inside the bracket. Terms are separated by + and – signs. See Topic 2 Sub-topic 1 Unit 1 for more on terms.

Remember to apply the exponent outside the bracket to ever part of the term inside the bracket.

The answer to the final question in the previous activity is a very interesting because it combines the three types of situations we come across when dealing with rational exponents and roots.

1. Some radicals or roots can be written as **rational numbers** (positive or negative whole numbers or fractions) e.g. or .
2. Some radicals or roots cannot be written as rational numbers and it is best to leave them in radical form e.g. or . It is best to leave these in radical form because they are **irrational** numbers. A calculator can only work out a rough approximation of their value. These numbers cannot be written as a fraction and, when written as decimals, go on forever and ever and never repeat themselves.

NOTE: If you interested in exploring the most famous irrational number, , to learn more about what irrational numbers are like then watch the video called [Pi](https://www.youtube.com/watch?v=yJ-HwrOpIps) (09:40) and the video called [Mile of Pi](https://www.youtube.com/watch?v=0r3cEKZiLmg) (06:30) on Youtube.

1. Some radicals seem to not even be real numbers at all e.g. or . We call these **non-real** numbers. The weird thing is that they are actually every bit as real as the real numbers but kind of exist in another dimension.

NOTE: If you are interested in exploring non-real numbers a bit more, watch the video called [Imaginary Numbers Are Real [Part 1: Introduction]](https://www.youtube.com/watch?v=T647CGsuOVU) (05:45) on YouTube and enjoy going down the rabbit hole.

We call radicals that are irrational numbers **surds**. Watch the video called [What are surds?](https://www.fuseschool.org/topics/179#contents/1319) (04:20) to find out more.

(<https://www.fuseschool.org/topics/179#contents/1319>)

There is one thing that we need to correct about the previous video. It said that all surds are square roots. This is not true. Any radical (square, cube or nth root) that is an irrational number is a surd.

### Activity 3: Surds

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the basics of working with surds. |
| Stopwatch | Suggested Time You will need about 20 minutes. |
| Pencil | What You Will Need  * A pen * Some blank paper or a notebook |

#### Tasks

1. Consider .
2. Is this a perfect square? In other words, is there a whole number that we can multiply by itself to get 98?
3. Write 98 as a product of its prime numbers. Start by dividing 98 by the smallest prime number which goes in exactly (which is 2) and keeping dividing the result by the smallest prime number that divide exactly each time, until the answer is 1. Remember prime numbers are numbers that can only be divided exactly by 1 and themselves e.g. 2, 3, 5, 7, 11, etc.
4. Is there a simpler way we can write now?
5. Write each of the following in the simplest form possible, leaving your answer with surds if necessary.

#### Guided Reflection

1. We are considering .
2. is not a perfect square like or .
3. We can write 98 as the product of prime factors like this.

The smallest prime that divides exactly into 98 is 2: .

Now we cannot divide 49 exactly by 2, or 3, or 5. The smallest prime that goes in exactly is 7: .

The smallest prime that goes into 7 exactly is 7:

Therefore, we can write 98 as .

1. This means that we can re-write as
2. Try answer all the parts of question 2 and then watch the video called [Square roots and real numbers](https://www.youtube.com/watch?v=BpBh8gvMifs) (10:20) to see all the worked solutions.

(<https://www.youtube.com/watch?v=BpBh8gvMifs>)

### Activity 4: Simplifying Surds

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the basics of working with surds. |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen * Some blank paper or a notebook |

#### Tasks

1. Simplify the following expressions as far as possible, leaving the answer in surd form if necessary.
2. Hint: Remember that therefore .

#### Guided Reflection

1. We had to simplify the given expressions.

We could also keep the expression written in root form.

(you should be able to see how to split 300 and 90 so that you can take a perfect square outside of the square root sign)

(now we can just multiply the numbers together and the surds together)

1. . We should first write 54 as the product of primes to see if there are any perfect cubes.

Therefore

1. . Start by writing the radicands as a product of primes.

|  |  |
| --- | --- |
|  |  |

(we know that the value of is fixed and so we can think of this as )

1. . Again, we should write each radicand as a product of primes.

|  |  |
| --- | --- |
|  |  |

Notice how we split the into so that we could take the perfect square of out of the square root.

1. .

|  |  |
| --- | --- |
|  |  |

1. . We need to start by simplifying what is inside the bracket to one term by subtracting the fractions.

(the LCD is )

For now, this is as far as we can simplify this question.

Let’s quickly summarise some of the rules we have developed for working with surds.

Watch these two videos for an excellent summary of what we know so far.

1. [Multiplying and dividing surds](https://www.fuseschool.org/topics/179#contents/1335) (02:08)

(<https://www.fuseschool.org/topics/179#contents/1335>)

1. [Adding and subtracting surds](https://www.fuseschool.org/topics/179#contents/1382) (01:57)

(<https://www.fuseschool.org/topics/179#contents/1382>)

In the last question in the previous activity, we found that we could not get very far. However, there is a technique, called **rationalising the denominator**, that we could use to simplify this expression further. Let’s see how it works.

Watch the video called [Rationalising the Denominator](https://www.fuseschool.org/topics/179#contents/1561) (03:40).

(<https://www.fuseschool.org/topics/179#contents/1561>)

### Activity 5: Rationalise the Denominator

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the basics of working with surds. |
| Stopwatch | Suggested Time You will need about 20 minutes. |
| Pencil | What You Will Need  * A pen * Some blank paper or a notebook |

#### Tasks

1. Simplify the following expressions as far as possible.
2. Hint: There is a cube root in the denominator.

#### Guided Reflection

1. We were asked to simplify the expressions.

(remember that we have to multiply by the equivalent of 1 so that we do not change the value of the fraction which is why we need to multiply by )

We have removed any irrational numbers from the denominator. We have *rationalised* the denominator.

is so much simpler looking than !

(here we need to multiply by because there was a cube root in the denominator - )

(remember that we need to multiply the denominator by in order to rationalise it)

If this last step does not make much sense, then take a look at Topic 2 Sub-topic 1 Unit 2. Also, for more insights into why , have a look at Topic 2 Sub-topic 1 Unit 2 and Unit 3 Activity 4.

## Unit 3: Logarithms

#### Learning Outcomes

By the end of this unit, you should be able to:

#### Introduction

Text goes here…