Topic 1: Number and Number Relationships

NASCA Mathematics

[Sub-Topic 2: Sequences and Series 3](#_Toc3991031)

[Unit 1: Arithmetic Sequences 4](#_Toc3991032)

[Activity 1: What is an Arithmetic Sequence 4](#_Toc3991033)

[Activity 2: Playing with Arithmetic Sequences 10](#_Toc3991034)

[Activity 3: Solving Arithmetic Sequence Problems 14](#_Toc3991035)

[Unit 2: Quadratic Sequences 19](#_Toc3991036)

[Activity 1: Blue Tiles 19](#_Toc3991037)

[Activity 2: More About Quadratic Sequences 22](#_Toc3991038)

[Activity 3: Quadratic Sequences in Real Life 25](#_Toc3991039)

[Unit 3: Geometric Sequences 26](#_Toc3991040)

[Activity 1: Bacteria 26](#_Toc3991041)

[Activity 2: Epidemics and other Geometric Sequences 29](#_Toc3991042)

[Unit 4: Arithmetic Series 33](#_Toc3991043)

[Activity 1: The Sum of an Arithmetic Sequence 33](#_Toc3991044)

[Activity 2: Arithmetic Series 36](#_Toc3991045)

[Activity 3: The Power of the Sigma and more Series 39](#_Toc3991046)

[Unit 5: Geometric Series 43](#_Toc3991047)

[Activity 1: Introducing Geometric Series 43](#_Toc3991048)

[Activity 2: More Geometric Series 45](#_Toc3991049)

[Activity 3: To Infinity and Beyond 47](#_Toc3991050)

[Activity 4: Infinite Geometric Series 50](#_Toc3991051)

[Sub-Topic 3: Finance 54](#_Toc3991052)

[Unit 1: Simple and Compound Growth and Decay 54](#_Toc3991053)

[Activity 1: Simple and Compound Interest 55](#_Toc3991054)

[Activity 2: Simple and Compound Growth 62](#_Toc3991055)

[Activity 3: Simple and Compound Depreciation 65](#_Toc3991056)

[Activity 4: Straight-Line and Reducing Balance Depreciation 71](#_Toc3991057)

[Activity 5: Nominal and Effective Rates 74](#_Toc3991058)

[Activity 6: Timelines 76](#_Toc3991059)

[Unit 2: Annuities, Loans and Investments 80](#_Toc3991060)

[Activity 1: Save Regularly 80](#_Toc3991061)

[Activity 2: Future Value Annuities 83](#_Toc3991062)

[Activity 3: Sinking Funds 87](#_Toc3991063)

[Activity 4: Present Value Annuities 90](#_Toc3991064)

[Activity 5: Loans 94](#_Toc3991065)

[Activity 6: Financial Analysis 100](#_Toc3991066)

# Sub-Topic 2: Sequences and Series

Mathematics is really all about finding patterns. Patterns are what help us use our knowledge of the past to predict what will most likely happen in the future. When engineers build a bridge, for example, they apply their knowledge of patterns to build a bridge that they are sure will carry the load it is designed to.

Scientists who study the human brain say that it is really a pattern recognising machine. Every time you see a face, your brain recognises the patterns – the location of the eyes, the placement of the mouth, where the ears are. The human face is really just a set of complex patterns. This is what allows us programme computers to be able to recognise faces in general and one person’s face in particular.

We are so good at recognising patterns that we know immediately what is in this picture.



Image source: <http://res.publicdomainfiles.com/pdf_view/2/13493562019700.png>

It is relatively easy to programme a computer to recognise the patterns in this shape as that of a bicycle. But getting a computer to recognise these as bicycles is much, much, much harder!



Image source: <https://www.publicdomainpictures.net/pictures/250000/nahled/bicycle-rear-wheels.jpg>

Our brains being such amazing pattern recognising machines is one of the reasons why doing maths is such a human activity. In many ways, Mathematics is our attempt to formalise our instinctive ability to recognise patterns and to put this ability to use to…programme computers to also recognise patterns.

All of Mathematics is about recognising and describing patterns, but in this sub-topic, we are going to look at some different kinds of patterns directly.

## Unit 1: Arithmetic Sequences

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Recognise arithmetic patterns;
2. Explain what a sequence, term, general term and common difference is;
3. Derive the formula for the general term of an arithmetic sequence;
4. Find the first term, general term and common difference of an arithmetic sequence; and
5. Solve problems involving arithmetic sequences.

#### Introduction

The term “Arithmetic Sequence” sounds very fancy, complicated and formal. In reality, we are surrounded by arithmetic sequences and use have already used them many times in your life. Every time you count, you are using an arithmetic sequence. Every time you have changed the quantities of ingredients in a recipe to make more or less of the dish, you have used an arithmetic sequence.

The best way to learn what an arithmetic sequence is to do an investigation.

### Activity 1: What is an Arithmetic Sequence

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will help understand what an arithmetic sequence is. |
| Stopwatch | Suggested TimeYou will need about 40 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Imagine you are a kid’s party planner, planning a party and you need to work out how many tables you need for the guests. The plan is to use square tables that seat four kids and to join these up into one long table like this.



In this case, two tables can seat 6 kids.

1. Draw out this same pattern to see how many kids one, two, three, four and five tables joined together like this can seat.
2. Create and complete a table like this.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Tables |  |  |  |  |  |  |
| Kids |  |  |  |  |  |  |

1. How many kids could you seat if you used ten tables?
2. How many tables would you need to be able to seat 28 kids?
3. What is the number of kids we need to add each time we add another table?
4. What is the general rule that creates the sequence of number of kids that can be seated for each table added?
5. You also need to create some decorations for the party. One of your ideas is to create straw decorations with a square pattern like this.



1. Complete a table showing the number of squares in the pattern and the number of straws needed to help you figure out what the general rule is that determines how many straws you need for any number of squares in the pattern?
2. How many straws will you need to make a pattern of 8 squares.
3. How many squares can you make if you have 37 straws?
4. If each straw is 22cm long and you have 328 straws, what is the maximum length of the square pattern decoration that you can make?
5. You also decide to use the same technique to make an alternating triangle pattern using coloured straws threaded through with string. The pattern you want to create looks like this.



1. The pattern above has three triangles. Complete a table showing the number of triangles in the pattern and the number of straws needed to help you figure out what the general rule is that determines how many straws you need for any number of triangles in the pattern?
2. Work out how many straws you will need for a pattern consisting of eleven triangles.
3. How many straws will you need for a pattern consisting of 22 triangles?

#### Guided Reflection

1. As the party planner, you need to work out how many tables you will need to seat all the kids.
2. If we draw out the pattern for one, two, three, four and five tables joined together we get the following:











1. Here is the completed table based on our drawings. It looks like the pattern is that we add two kids for each table so for six tables we will have kids. We can draw a picture to confirm this.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Tables |  |  |  |  |  |  |
| Kids |  |  |  |  |  |  |

1. Each extra table allows us to seat two extra kids. We know that five tables lets us seat twelve kids. Therefore, an extra five tables will allow us to seat an extra kids i.e. kids in total.
2. We know that one table lets us seat 4 kids and that each extra table adds two to this number. Therefore, if we have 28 kids, the first table will seat four kids, meaning that the other tables need to seat 24 kids. This means that we will need extra tables i.e. 13 tables in total.
3. For each table, we need to add two kids.
4. The first table seats 4 kids. Each extra table seats an additional two kids.

For the **first** table second table we have four kids.

For the **second** table we have kids.

For the **third** table we have kids.

For the **fourth** table we have kids.

For the **fifth** table we have kids.

It looks like we have to multiply 2 by **one less** than the table number. Therefore, for the nth table, we will have kids.

1. We used straws to make a pattern of squares.
2. Here is a table showing the number of straws needed for different numbers of squares.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Squares |  |  |  |  |  |  |
| Straws |  |  |  |  |  |  |

The **first** square needs straws.

**Two** squares need straws.

**Three** squares need straws.

**Four** squares need straws.

**Five** square need straws.

Once again, we have to multiply the number of additional straws by **one less** than the number of squares we want. Therefore, for the nth square, we will need straws.

1. To make eight squares we will need straws.
2. If we have straws, we know that . Therefore

Therefore, we can make twelve squares.

**Note**: If you need help solving the equation above, you should have a look at Topic 2, Sub-Topic 2, Unit 1. There will be many equations like this to solve in this sub-topic.

1. We know we have 127 straws, so let’s find out how many squares we can add to our pattern.

So, we will have 42 squares in our pattern and the width of each square is 22cm. This means that the whole pattern will be or long.

1. This time, we are making a decoration with triangles.
2. Here is a table showing the number of straws needed for different numbers of squares.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Triangles |  |  |  |  |  |  |
| Straws |  |  |  |  |  |  |

The **first** triangle needs straws.

**Two** triangles need straws.

**Three** triangles need straws.

**Four** triangles need straws.

**Five** triangles need straws.

Once again, we have to multiply the number of additional straws by **one less** than the number of triangles we want. Therefore, for the nth triangle, we will need straws.

1. Eleven triangles will need straws.
2. Twenty-two triangles will need straws.

In each case in the previous activity we had a **sequence** of numbers (the number of kids that could be seated, the number of straws needed to make a square or triangle pattern) that followed a special **order** or **pattern**.

In all cases, we had to add a specific number to each term in the sequence to generate the next term in the sequence.

We call sequences like this **arithmetic sequences** and we call the number we need to add to each term to create the next term the **common difference**. It is called the common difference, because the difference between any two successive terms is always the same. It is common.

We call the numbers in these sequences **terms** and write them like this:

or sometimes

 means “term one” or the “first term”. means “term n” or the “**general term**”. Sometimes you will see for the general term.The letter really does not matter.

We also discovered that we could write down a general rule to describe each of the sequences in the previous activity.

* Kids at tables:
* Straws needed for square pattern:
* Straws needed for triangle pattern:

Each of these rules could be used to work out the value of any term in the sequence. For example, if we wanted to work out the value of in the kids at tables sequence we would just replace with and calculate .

Can you see that each of these general rules is really the same? Each one has added to where is the common difference.

So, in general, the general term of an arithmetic sequence is always given by this formula:

where is the first term.

Before tackling the next activity, learn more about sequences in general by watch the video called [Sequences intro](https://www.khanacademy.org/math/algebra/sequences/introduction-to-arithmetic-squences/v/explicit-and-recursive-definitions-of-sequences) (08:15).

(<https://www.khanacademy.org/math/algebra/sequences/introduction-to-arithmetic-squences/v/explicit-and-recursive-definitions-of-sequences>)

### Activity 2: Playing with Arithmetic Sequences

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will help consolidate your understanding of arithmetic sequences. |
| Stopwatch | Suggested TimeYou will need about 50 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Have a look at the following sequences and write down the next three terms in the sequence. Then decide if the sequence is arithmetic or not. If the sequence is arithmetic, write down the formula for the general term.
2. The general term of a sequence is .
3. What is the first term?
4. What are and ?
5. What is the common difference?
6. Find the general term of each arithmetic sequence.
7. The third term is 16 and the sixth term is 37. **Hint**: Work out the common difference based on the total difference between the third and sixth terms.
8. The fourth term is and the tenth term is .
9. and .
10. The first three terms of an arithmetic sequence are given - .
11. Find the value of . **Hint**: Because this is an arithmetic sequence, we that that
12. What is the formula for the general term?
13. What is ?
14. If an arithmetic sequence is , how many terms does it have?
15. The arithmetic mean of and is . Determine the value of in terms of . **Hint**: The arithmetic mean of and is and therefore form an arithmetic sequence.

#### Guided Reflection

1. We were given a number of sequences to investigate.
2. The next three terms in the sequence are .

Therefore, this is an arithmetic sequence with a first term of 1 and a common difference of 1.

1. The next three terms in the sequence are .

Therefore, this is an arithmetic sequence with a first term of 1 and a common difference of 2.

1. The next three terms in the sequence are .

Therefore, this is an arithmetic sequence with a first term of 1 and a common difference of 4.

1. The next three terms in the sequence are .

Therefore, this is an arithmetic sequence with a first term of 5 and a common difference of -3.

1. The next three terms in the sequence are .

Therefore, this is not an arithmetic sequence because there is no common difference which generates each successive term.

The next three terms in the sequence are .

Therefore, this is an arithmetic sequence with a first term of and a common difference of .

1. The general term of the given sequence was .
2. The first term is .
3.
4. In each case, we need to find the general term, .
5. If and then

But there are three common differences needed to get from to . Therefore, .

If , then .

So .

1. If and then .

But there are six common differences needed to get from to . Therefore, .

If , then .

So .

1. If and then .

But there are six common differences needed to get from to . Therefore, .

If , then .

So .

**Note**: If you need help manipulating the algebra (terms with and in them), then you should have a look at Topic 2, Sub-topic 1.

1. We are told that the first three terms of an arithmetic sequence are .
2. Because this is an arithmetic sequence, we know that

So,

1. . But and .
2. The arithmetic sequence is . Therefore and .

So, .

We know that the last term is .

Therefore, we let and solve for .

There are 48 terms in the sequence.

1. We are told that the arithmetic mean of and is . Therefore, we know that

 (Now we can solve for )

As we saw in Activity 1, there are many times when real life problems can be solved using arithmetic sequences. This next activity will explore some more examples.

### Activity 3: Solving Arithmetic Sequence Problems

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you an opportunity to solve some real-life problems using arithmetic progressions. |
| Stopwatch | Suggested TimeYou will need about 60 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. The sum of the first three consecutive terms of an arithmetic sequence is 12 and the product of the first two is eight. What is the sequence?
2. A cyclist needs to train for a 250km race. He decides to train every second day by riding 15km further each successive training day so that on his final training day he rides 250km. If he has four weeks in which to train, how long does his first ride need to be?
3. A consignment of 20 pieces of steel has been supplied to an engineering company. The quality inspector realises that the 11th item is 9 times the first item in mass (in kg), while the 7th item is one unit more than two times the mass of the third item. She is given information that the difference between any two consecutive masses is a constant. Find the mass of the 20th piece.
4. If a clock strikes once at 1'o clock, twice at 2'o clock and so on. How many times will it strike in a 24-hour period?
5. A gardener plans to construct a trapezoidal shaped structure in his garden. The longer side of trapezoid needs to start with a row of 97 bricks. Each row must be decreased by 2 bricks on each end and the construction should stop at 25th row. How many bricks are in the last row?
6. The sum of R1,000 is deposited into an account which earns simple interest at 8% per year. How much is in the account after 30 years?
7. A construction company will be penalized each day of delay in construction for bridge. The penalty will be R4,000 for the first day and will increase by R10,000 for each following day. Based on its budget, the company can afford to pay a maximum of R165,000 toward penalties. Find the maximum number of days by which the completion of work can be delayed.

#### Guided Reflection

1. We are told that the sum of the first three numbers in an arithmetic sequence is 12 and the that product of the first two numbers is eight.

Because we know it is an arithmetic sequence, we know that there is a common difference between each term. Therefore, if , and . But

We also know that

So, we have two equations:

 (1)

 (2)

From (1): (3)

Substitute (3) into (2):

Sub into (3):

Therefore, the sequence is

**Note**: We needed to solve two equations simultaneously in this question. If you need help solving simultaneous equations, see Topic 2 Sub-topic 2 Unit 5.

1. We know that the common difference of this arithmetic progression is 15 and that the last term is 250. We also know that there are terms in the sequence.

Therefore .

His first ride needs to be 55km.

1. We know this is an arithmetic sequence because the difference between any two consecutive masses is constant. Let . Then and .

Let the constant difference be . This means that and . We have two equations:

 (1)

 (2)

Substitute (2) into (1):

Substitute into (2):

The sequence is . Therefore,

kg

1. Our basic arithmetic sequence is . If we add up each of these terms, we will find how many times that clock strikes in 12 hours.

So, .

But we were asked how many times it strikes in a 24-hour period. This will be double the 12-hour period i.e. 156 times.

When we have an arithmetic sequence where all the terms are added, we call this an **arithmetic series** but more on these later.

1. The first row has 97 bricks. Therefore and . So,
2. The starting amount, or first term, is 1000. At the end of year 1, 8% interest is paid which is R80. So, at eth end of year 1, the account has R1,080 in it. At the end of year 2, interest is paid again but because it is simple interest, it is only generated on the original R1,000 i.e. another R80. So, now there is R1,160 in the account and so on like this.
* Year 0: R1,000
* Year 1: R1,000 + R80 = R1,080
* Year 2: R1,000 + R80 + R80 =R1,000 + 2 x R80 = R1,160
* Year 3: R1,000 + R80 + R80 + R80 = R1,000 + 3 x R80 = R1,240

Therefore, we have an arithmetic sequence where and . Except this time, the sequence starts at so instead of we will have .

Therefore, .

There will be R3,400 in the account after 30 years.

1. If the penalty on day 1 is R4,000 and it increases by R10,000 for each successive day, we have an arithmetic sequence where and . However, now we have to find out how many terms in the sequence will have a sum of or less.

The sequence is

The sum of three terms is .

The sum of six terms is .

The sun of five terms is .

Therefore, the company can only afford to be five days late.

Like question 4, we need to find the sum of the terms of an arithmetic sequence. There is an easier way we can do this which we will cover in Unit 4.

## Unit 2: Quadratic Sequences

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Recognise quadratic sequence patterns;
2. Derive the formula for the general term of a quadratic sequence;
3. Find the values of , and for a quadratic sequence; and
4. Solve problems involving quadratic sequences.

#### Introduction

In the previous unit, we explored arithmetic sequences but there is another kind of sequence which is somewhat similar that is also very useful in solving some everyday problems. Without spoiling the surprise, let’s dive into the first activity.

### Activity 1: Blue Tiles

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will introduce you to quadratic sequences. |
| Stopwatch | Suggested TimeYou will need about 20 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. A tiler has been asked to create the following pattern of tiles down the wall next to a long escalator leading to a refurbished subway station in New York City.



1. Write down the sequence for the number of white tiles.
2. What kind of sequence does the number of white tiles follow?
3. Write down the sequence for the number of blue tiles. How many blue tiles will there be in the 5th and 6th patterns in the sequence? Draw the patterns to make sure. Does this sequence follow an arithmetic progression?
4. Create a table like this of this sequence

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Pattern |  |  |  |  |  |  |
| Blue tiles |  |  |  |  |  |  |

What are the differences between each successive term in the blue tile sequence? What do you notice about these differences? What is the difference between each of the differences?

1. What general rule does the sequence of the number of blue tiles follow? Use this rule to work out how many blue tiles there will be in the 11th pattern in the sequence.
2. How many tiles in total will the 12th pattern in the sequence have?

#### Guided Reflection

1. We were given a series of tile patterns.
2. The sequence of the number of white tiles in the pattern is .
3. It looks like the number of white tiles follows an arithmetic sequence where and .
4. The sequence of the number of blue tiles, including the 5th and 6th pattern, is .

Here are the drawn patterns to confirm this.



This is not an arithmetic sequence because there is no common difference between the terms.

1. Here is the completed table.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Pattern |  |  |  |  |  |  |
| Blue tiles |  |  |  |  |  |  |

The differences between each successive term in the sequence increase by two each time. The differences between each successive difference is two.

1. A general rule for this pattern is that the nth term is . Therefore the 11th pattern in the sequence will have blue tiles in it.
2. There are two ways we can think of this. Either we can work out the number of white and blue tiles separately and then add these together, or we can recognise a similar pattern in the total titles as exists for the blue tiles.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Pattern |  |  |  |  |  |  |
| Blue tiles |  |  |  |  |  |  |

Therefore, the general term of this pattern will be . So, the 12th pattern in the sequence will have tiles in it.

In the previous activity, we discovered two versions of the same kind of pattern. In both cases, the pattern involved squaring and, for this reason, they are called **quadratic sequences**. In Topic 2, we will learn about quadratic equations and quadratic functions. These are equations which have an in them.

We also saw that quadratic sequences do not have a common difference but that the difference between the difference of each successive term is common. We call this the **second difference**.

Remember, this table for the blue tile sequence? Here we can see that the second difference is constant and is two.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Pattern |  |  |  |  |  |  |
| Blue tiles |  |  |  |  |  |  |

First Difference

Second Difference

We also discovered that the general term for the blue tile sequence was and for the total tile sequence, it was .

We can expand both of these expressions as follows:

and

Note: If these expansions are not making sense to you, visit Topic 2, Sub-topic 1, Unit 2.

Both expressions have a general form of

where , and are just numbers. If you know about quadratic equations, you will recognise this as the same general equation.

Let’s investigate these quadratic sequences in a little more detail.

### Activity 2: More About Quadratic Sequences

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will explore more details about quadratic sequences. |
| Stopwatch | Suggested TimeYou will need about 20 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. In the previous activity, we discovered two different quadratic sequences:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Pattern |  |  |  |  |  |  |
| Blue tiles |  |  |  |  |  |  |

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Pattern |  |  |  |  |  |  |
| Blue tiles |  |  |  |  |  |  |

1. If the general form of a quadratic sequence is , write down the values of , , and for each expression.
2. Find an expression containing , , and that gives the value of the first term for each sequence.
3. Test your expression in b) on the quadratic sequence .
4. Find an expression that gives the value of the second term in each sequence by combining the values of , , and . Hint: the expression contains .
5. Find an expression that gives the value of the third term in each sequence by combining the values of , , and . Hint: the expression contains .
6. What are the first differences between the expressions for and and between and ?
7. What is the second difference between these expressions?
8. Complete this diagram showing the relationships between the first four terms of a quadratic sequence, the first differences and the second difference.



1. Given the quadratic sequence :
2. What is the second difference?
3. Using your expression for second difference discovered in question 1, what is the value of for this sequence.
4. What are the values of and for this sequence?
5. What is the expression for the general term of this sequence?
6. Complete the quadratic sequence by filling in the missing term:

#### Guided Reflection

1. We discovered two different quadratic sequences - and .
2. The general form of a quadratic sequence can be written as . Therefore, the values of , , and for

 are , , and

 are , , and

1. We need to find an expression for the first term of each sequence in terms of , , and .

. might work. Let’s see if it works for the other sequence.

. .

It looks like the expression for the first terms of a quadratic sequence is given by .

1. The expression is .

Therefore for a quadratic sequence.

1. We need to find an expression for the second term of each sequence in terms of , , and where the expression contains .

. Let’s consider the fact that we are dealing with term 2. Our expression can be written as . We have just made the values of , , and explicit in the expression. Therefore, it stands to reason that for term 2 when the value of , the expression will be . In other words, the expression is which can be written as .

Let’s check: .

Therefore

1. Using the same logic is above we can say that .
2. .

.

1. .
2. Here is the completed diagram:



Image source: Everything Maths Gr 11 (pg 91)

1. We are told that is a quadratic sequence.
2. We know that and .

 (1)

 (2)

We need to solve these equations simultaneously. We can subtract (1) from (2):

Therefore, the second difference is 10.

1. We know that . Therefore, .
2. and . Therefore, .

We know that where and . Therefore .

1. . So, our sequence is given by the expression .
2. We basically have to find the four term of this quadratic sequence: .

We know that and . Therefore, the second difference, .

 but . Therefore, .

So, the general term of the sequence is given by Therefore,

### Activity 3: Quadratic Sequences in Real Life

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you an opportunity to use quadratic sequences to solve some real-life problems |
| Stopwatch | Suggested TimeYou will need about 40 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Everything Maths Gr11 (pg 101 – Q15)
2. Everything Maths Gr11 (pg 96 – Ex 4)
3. Everything Maths Gr11 (pg 102 – Q17)
4. Everything Maths Gr11 (pg 99 – Q2 c, e, f, g)

## Unit 3: Geometric Sequences

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Recognise geometric sequence patterns;
2. Derive the formula for the general term of a geometric sequence;
3. Find the values of and for a geometric sequence; and
4. Solve problems involving geometric sequences.

Introduction

So far in this sub-topic, we have explored two different kinds of sequences – the arithmetic sequence and the quadratic sequence. But there is a third kind of sequence commonly found in real life situations.

### Activity 1: Bacteria

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will introduce you to geometric sequences. |
| Stopwatch | Suggested TimeYou will need about 20 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. One of the characteristics of bacteria (which are single celled organisms) like E. coli that can make them potentially so harmful to humans is their ability to reproduce very quickly from a single individual bacterium. When bacteria reproduce, they do so by simply splitting into two cells. These two cells can then each split into two more cells and so on. Here is a picture of this kind of reproduction.



Image source: <https://cdn.pixabay.com/photo/2013/07/13/10/15/bacteria-156867_960_720.png>

1. Complete this table showing the number of bacteria for each step in the sequence.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Step |  |  |  |  |
| Number of bacteria |  |  |  |  |

1. Is this an arithmetic or quadratic sequence? Why or why not?
2. What will the 5th and 6th terms in the sequence be?
3. Work out an expression that will help you figure out what the 12th term in the sequence is.
4. If these bacteria divide like this once every 8 hours, after how many hours will there be more than 1 million bacteria?
5. How long will it take for there to be more than 10 million bacteria? How many days is this?

#### Guided Reflection

1. We are dealing with multiplying bacteria.
2. Here is the completed table.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Step |  |  |  |  |
| Number of bacteria |  |  |  |  |

1. This is not an arithmetic sequence. An arithmetic sequence has a common difference between the terms. In this sequence, there is no single number that we add to each term to get the next term in the sequence.

It is also not a quadratic sequence, although it does look quite similar. If we try to find the constant second difference, we see that there isn’t one.



1. If we follow the pattern that each term is double the previous term then the fifth term will be 16 and the sixth term will be 32.
2. It looks like the sequence is generated by where

Therefore,

1. We need to find the first term that is greater than 1 million. We can do this by trial and error, or we can solve the equation .

Let’s use trial and error.

. If we double this, we only get to about .

. If we double this, we only get to about

. If we double this, we get to about which is the first term we have found greater than 1 million.

.

If it takes 8 hours for each division, it will take a total of hours for there to be more than 1 million bacteria.

1. We can keep using trial and error to find the first term greater than 10 million.

. Almost there.

.

The 25th term is the first term greater than 10 million. Therefore, it will take hours for there to be more than 10 million bacteria. This is days and 8 hours.

From the activity above, we can see that sequences that involve doubling of terms to get the next term, increases very quickly. We call these kinds of sequences **geometric sequences**.

Where arithmetic sequences have a common difference and quadratic sequences have a second common difference, geometric sequences have a **common ratio**.

In the geometric sequence above, we can see that . If we divide any term by the previous term, the answer is always the same.

The general form of a geometric expression is

where is the common ratio and is the first term in the sequence.

For example, the sequence is a geometric progression because . It has a common ratio of . In other words, to generate the next term in the sequence, you have to multiply the previous term by .

### Activity 2: Epidemics and other Geometric Sequences

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you an opportunity to explore other kinds of geometric sequences. |
| Stopwatch | Suggested TimeYou will need about 60 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Epidemics very often follow a geometric sequence. For example, a flu epidemic was found to follow the sequence where was the number of days and was the number of new infections on any given day.
2. How many new people were infected with the flu virus on day 10?
3. On which day were 16,384 new people infected?
4. If the 5th term of a geometric sequence is 1,875 and the common ratio is 5, what are the first three terms?
5. What is the first term and the common ratio of the geometric sequence where the 4th term is and the 7th term is and ?
6. The nth term of a sequence is given by . Write down the first three terms. What kind of sequence is this?
7. Find if form a geometric sequence. What are the possible values of ?
8. Consider the sequence . The first three terms form an arithmetic sequence while the last three terms form a geometric sequence. What are the values of and if both are positive integers?
9. If the geometric mean is the number, , that must be inserted between and so that form a geometric sequence, what three geometric means must be inserted between and .
10. What is the 29th term of the sequence ? Is this an arithmetic, quadratic or geometric sequence?

#### Guided Reflection

1. The epidemic followed the sequence .
2. . There were 1,024 new infections on day 10.

There were 16,384 new infections on day 14.

**Note**: To solve for we had to solve an exponential equation. If you need more help with exponential equations, work through Topic 2 Sub-topic 2 Unit 3.

1. The 5th term of a geometric sequence is 1,875 and the common ratio is 5. Therefore,

Therefore, . So , and .

1. We know that and .

We know that and that and . So, we can say that

1. The nth term of a sequence is given by .

.

This is a geometric sequence because there is a common ratio of .

1. If for a geometric sequence, then

.

If , then .

If , then .

1. The first three terms of form an arithmetic sequence. Therefore, we know that

 (1)

The last three terms form a geometric sequence. Therefore, we know that

 (2)

Substitute (1) into (2):

 or

But we are told that is a positive integer. Therefore, is the only valid solution.

Substitute into (1):

.

1. Let’s take as the first term in the sequence. If we need to insert three terms between and , then will be the 5th term in the sequence.

We know that because .

Therefore,

So, the sequence is .

## Unit 4: Arithmetic Series

#### Learning Outcomes

By the end of this unit, you should be able to:

1. xxx

Introduction

Remember these questions from Unit 1?

If a clock strikes once at 1'o clock, twice at 2'o clock and so on. How many times will it strike in a 24-hour period?

A construction company will be penalized each day of delay in construction for bridge. The penalty will be R4,000 for the first day and will increase by R10,000 for each following day. Based on its budget, the company can afford to pay a maximum of R165,000 toward penalties. Find the maximum number of days by which the completion of work can be delayed.

In both cases, we had to add up the terms in an arithmetic sequence. For example, in the clock question, we found that the sequence was and that the sum of this sequence was . We had to add the numbers up manually.

It turns out, however, that adding up the terms of an arithmetic sequence can be very useful in solving many everyday problems, especially ones that relate to money. But adding up the numbers in a sequence can be long and tedious. Remember, Mathematicians are lazy. Let’s see if we can find a shortcut.

### Activity 1: The Sum of an Arithmetic Sequence

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will introduce you to arithmetic series. |
| Stopwatch | Suggested TimeYou will need about 20 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. A clock strikes once at 1'o clock, twice at 2'o clock and so on.
2. Complete the third row of this table which gives you the total number of times the clock strikes in a 12-hour period. means the sum of the number of strikes in 12 hours.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Hour |  |  |  |  |  |  |  |  |  | 0 |  |  |
| Number of strikes |  |  |  |  |  |  |  |  |  | 0 |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |

1. Now complete the 4th row by writing this same expression in reverse i.e. starting with the number of times the clock will strike at 12’o clock. Will the expressions in rows three and four have the same answer?
2. What do you notice about the sum of the numbers in each column for the two expressions for ? Write down an expression in the last row of the table for .
3. Write down a simple expression for .
4. Use your expression for above to write a simple expression for .
5. We know that the sum of strikes for a 12-hour period is 78. Does your expression for ?
6. Write a general expression for the sum of the first terms of this sequence, in terms of and .
7. If we know that , write an expression for in terms of , and .
8. Use this expression to work out the sum of the first 500 even numbers.

#### Guided Reflection

1. We are told that a clock strikes once at 1'o clock, twice at 2'o clock and so on.
2. Here is the table with row three completed.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Hour |  |  |  |  |  |  |  |  |  | 0 |  |  |
| Number of strikes |  |  |  |  |  |  |  |  |  | 0 |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |

1. Here is the table with row 4 completed.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Hour |  |  |  |  |  |  |  |  |  | 0 |  |  |
| Number of strikes |  |  |  |  |  |  |  |  |  | 0 |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |

Both expressions for will have the same answer.

1. The total of each column is . Here is the final row completed.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Hour |  |  |  |  |  |  |  |  |  | 0 |  |  |
| Number of strikes |  |  |  |  |  |  |  |  |  | 0 |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |

1. . Yes,
2. . Therefore
3. The sequence of even numbers is . Therefore, and .

We also know that

Therefore,

In the previous activity, we found an expression that allows us to find the sum of any number of terms of an arithmetic sequence. This expression is

We call this sum of the first terms of an arithmetic sequence an **arithmetic series**.

If an arithmetic sequence is , then the arithmetic series will be .

Many arithmetic sequences do not have a specific last term and go on forever. Such sequences are called **infinite arithmetic sequences** and the sum of **infinite arithmetic series** is also infinite i.e. there is no definite answer. The best we can say is that they sum to positive or negative infinity.

Arithmetic sequences where there are a finite number of terms are called **finite arithmetic sequences** and the sum of finite series do have definite answers.

Because infinite series do not give us definite answers, they tend not to be very useful in everyday situations.

### Activity 2: Arithmetic Series

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will allow you to practice what you know about arithmetic series. |
| Stopwatch | Suggested TimeYou will need about 20 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Find the sum of the first 100 natural numbers.
2. Find the sum of the second 100 natural numbers.
3. Find the sum of the first 30 terms of the sequence
4. Find the sum of the series
5. The sum of the second and third terms of an arithmetic sequence is equal to zero and the sum of the first 36 terms of the series is equal to 1,152. Find the first three terms in the series.

#### Guided Reflection

1. The sequence of natural numbers is an arithmetic sequence where and . Therefore .
2. To work out the sum of the second 100 natural numbers, we need to work out the sum of the first 200 natural numbers and then subtract the sum of the first 100 natural numbers.

Therefore, the sum of the second 100 natural numbers is .

1. The given sequence is We first need to check that this is an arithmetic sequence.

 and and

Therefore, there is a common difference and and .

So, .

1. The series given is . First, we need to check that this is an arithmetic series.

 and

Therefore, this is an arithmetic series with and .

Before we can work out the sum, we need to know what term is equal to 123.

. But and .

We need to find .

.

1. We are told that the sequence is arithmetic, that and that .

 (1)

 (2)

Substitute (1) into (2):

Substitute into (1)

So, the first three terms are .

The expression is not the only way we have of writing the sum of a given number of terms in a sequence. Other method that you will often see is called **Sigma notation** because it makes use of sigma, the Greek capital letter “S” for “sum”.

If we have the sequence , we know that we can express the sum of the series as .

But we can also express it as

Let’s unpack what everything means.



This means that

So,

We will revisit Sigma notation a few times in the next couple of units. For now, watch the video called [Summation notation](https://www.khanacademy.org/math/algebra2/sequences-and-series/alg2-sigma-notation/v/sigma-notation-sum) (04:30) for a good introduction.

(<https://www.khanacademy.org/math/algebra2/sequences-and-series/alg2-sigma-notation/v/sigma-notation-sum>)

### Activity 3: The Power of the Sigma and more Series

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you practice in using Sigma notation and answering more complicated arithmetic series questions. |
| Stopwatch | Suggested TimeYou will need about 40 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
* An Internet connection
 |

#### Tasks

1. You are given the series .
2. What kind of series is this? Why do you say this?
3. What is the first term and common difference?
4. What term in the sequence is 405?
5. Write this series in Sigma notation by starting the index at 1.
6. Write this series in Sigma notation by starting the index at 0.
7. What is the value of this sum?
8. Expand this series (write it out in full).
9. Determine the value of
10. Calculate the value of in the following:
11. The common difference of an arithmetic series is 3.
12. Calculate the values of for which the th term of the series is 93 and the sum of the first terms is 975.
13. Explain why there are two possible answers.
14. The sum of an arithmetic series is 100 times its first term, while the last term is 9 times the first term. Calculate the number of terms in the series if the first term is not equal to zero.

#### Guided Reflection

1. We are given the series .
2. This is an arithmetic series because there is a common difference between each of the terms.
3. and .
4. The sequence can be expressed as . Therefore, we need to find such that .

405 is the 200th term in the series.

1. We know that the sequence is represented by . So, the series, from term 1 to term 200, in Sigma notation would be

When , which is our first term.

When , which is our second term.

When , which is our last term.

1. If we start the index at zero, it means that we need to end the index at 199 and change the expression. Instead of having to subtract one from each index value, we can simply plug it in.

When , which is our first term.

When , which is our second term.

When , which is our last term.

Now watch the video called [Arithmetic series in sigma notation](https://www.khanacademy.org/math/algebra-home/alg-series-and-induction/alg-advanced-sigma-notation/v/writing-arithmetic-series-in-sigma-notation) (04:10) for another explanation of how to write this series in Sigma notation.

(<https://www.khanacademy.org/math/algebra-home/alg-series-and-induction/alg-advanced-sigma-notation/v/writing-arithmetic-series-in-sigma-notation>)

1. To work out the value of this series, we still need to use our arithmetic series formula. Sigma notation is just a notation. It does not give us a way to calculate the value of the series.
2. We need to write in expanded form. All we need to do is create a series of terms by incrementing the value of from 1 to 5.
3. We need to determine the value of . Remember, Sigma notation is just a notation. We still need to use our arithmetic series formula to work out the value.

Our first step is to use the Sigma expression to work out what , and are. The index starts at 2, so our series is .

We can see that and and we know that there must be 49 terms in the series (there are 49 numbers from 2 to 50).

.

1. We need to calculate the value of in .

We know that the sum of the first three terms of the series is 28 and we know that the terms in the series are . Therefore,

1. We are told that the common difference of an arithmetic series is 3 i.e. .
2. The th term of the series is 93. Therefore, we can say that

 (1)

Also, the sum of the first terms is 975. Therefore, we can say that

 (2)

Substitute (1) into (2):

 or

1. If we substitute these values into equation (1) above, we see that there are two possible first terms of and
2. We are told that the sum of an arithmetic series is 100 times its first term. Therefore, we can say that or that .

We are also told that the last term is 9 times the first term. Therefore, we can say that .

We need to calculate the number of terms in the series if the first term is not equal to zero.

 (1)

Substitute into

 or

But we were told that the first term is not equal to zero. Therefore and there are 20 terms in the series.

## Unit 5: Geometric Series

#### Learning Outcomes

By the end of this unit, you should be able to:

1. xxx

Introduction

Just like we can work out what the value of an arithmetic series is so we can also work out the value of a geometric series. As useful as arithmetic series are, geometric series are even more useful. Any time we need to work with investments or loans involving compound interest (basically all investments and loans), we are relying on geometric series.

If is a **geometric sequence**, then is a **geometric series**.

In the same way as we have finite and infinite arithmetic series, we also have finite and infinite geometric series. However, some infinite geometric series do give answers that are not infinity, as we shall see.

Watch the video called [Geometric series intro](https://www.khanacademy.org/math/algebra2/sequences-and-series/alg2-geometric-sequence-series/v/series-as-sum-of-sequence) (02:55) for a good introduction to geometric series.

(<https://www.khanacademy.org/math/algebra2/sequences-and-series/alg2-geometric-sequence-series/v/series-as-sum-of-sequence>)

Now let’s continue our investigation of geometric series with an activity.

### Activity 1: Introducing Geometric Series

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you an opportunity to explore different kinds of geometric series. |
| Stopwatch | Suggested TimeYou will need about 30 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. We are given the geometric sequence .
2. Write down the first five terms of the sequence.
3. Does this sequence have a last term?
4. What do you think the sum of this sequence i.e. what will the value of the series be?
5. Calculate . Can you use the arithmetic series formula to do this?
6. Complete this table for the given series.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |

1. What do you notice about the values in the first and second rows of the table? Work out what is and, hence the value of Confirm that this is the correct answer by working out manually.
2. Write an expression for for a geometric series in terms of , and . Remember that in our series and and .
3. Test this expression by evaluating using it and manually.

#### Guided Reflection

1. We were given the sequence .
2. The first five terms of this sequence are .
3. This sequence does not have a last term. It goes on forever.
4. The geometric series would have a value of negative infinity. Each term gets more and more negative and so, if we add all these ever-growing negative numbers together, the sum will be negative infinity.
5. The arithmetic series formula is . We do have a first term () but instead of a common difference (), we have a common ratio () and . Let’s see what happens if we use the formula.

We can see immediately that this is definitely not the answer. Let’s work out manually.

.

We cannot use the arithmetic series formula for geometric series.

1. Here is the completed table. Remember that .

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |

1. The values in the second row are the same as those in the first row one cell to the right. Therefore, if we were to calculate , we could redo the table like this.

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |

Doing this manually we get .

We get the same answer.

1. From the expression above we can see that

But , and . So, we can write our expression as follows:

 or .

1. Our series is given by .

The expanded series is . Therefore , and .

Evaluating the series manually we get .

Using our formula, we get .

It looks like our formula works!

In the previous activity we discovered a formula we can use to work out the value of a geometric series. However, like with the arithmetic series formula, this formula only works for finite series. In other words, it only works for series with a defined or limited number of terms. Series with an infinite number of terms cannot be evaluated with this formula because in this case, we do not know what value has. Remember infinity is not a number. It is a concept.

### Activity 2: More Geometric Series

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you practice in working with finite geometric series. |
| Stopwatch | Suggested TimeYou will need about 45 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Find the sum of the first 10 terms of .
2. You want to buy a car worth R500,000.00 on credit and agree the following terms with the seller. You will pay R2.00 the first month and then double the previous amount every month for 18 months.
3. Express the series in Sigma notation.
4. Calculate how much will you need to pay in total? Did you get a good deal?
5. The second term of a geometric series is 6 and the fifth term is 162. Find the sum of the first 9 terms.
6. Given , calculate the number of terms whose sum is .

#### Guided Reflection

1. The sequence we are given is This is a geometric sequence where and . Therefore,
2. If you start by paying R2.00 and double this every month for 18 months.
3. This series can be represented in Sigma notation as follows:

Or

1. , and

You paid R24,268 more than the value of the car but this is still far less than you would have paid if you had taken out an ordinary bank loan.

1. We are told that the second term of a geometric series is 6 and the fifth term is 162. Therefore, we know that and .

Therefore if , .

So,

1. We were given the series . This is the series We can see that and .

We want to find such that .

### Activity 3: To Infinity and Beyond

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will introduce you to the sum of infinity geometric series. |
| Stopwatch | Suggested TimeYou will need about 45 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. This was the sequence we dealt with in question 4 of the previous activity:
2. Calculate the sum to ten terms. Express the answer as a decimal fraction correct to five decimal places.
3. Calculate the sum to 15 terms. Express the answer as a decimal fraction correct to five decimal places.
4. Calculate the sum to 20 terms. Express the answer as a decimal fraction correct to five decimal places.
5. What do you notice about the sum as the number of terms gets bigger and bigger?
6. What do you think the sum to infinite terms will be?
7. Look back at your calculations for parts a) – c). What do you notice about the value of the part of the equation?
8. If we let get extremely big (infinite), what will the value of become?
9. What expression could we, therefore use to evaluate the sum of this series to infinite terms?
10. Look at these series:
11. Do these series get closer and closer to a specific finite answer if we let get extremely big? Try a few values for .
12. What kind of series are these? What is the difference between and that can explain why does not get closer and closer to a specific answer the larger we let get?

#### Guided Reflection

1. We are working with the series given by again.
2. The sum gets closer and closer to 8.
3. The sum to infinite terms will be equal to 8.
4. gets closer and closer to negative one as gets bigger and bigger.
5. If we let be an infinite number of terms .
6. The sum to an infinite number of terms would be .
7. We are given two more series to investigate.
8. If we let get extremely big, then does not close in on a single finite answer. Instead the sum just gets bigger and bigger.

If we let get extremely big, then does not close in on a single finite answer. Instead the sum just gets bigger and bigger, although it does not get bigger quite as quickly as the series above.

1. is a geometric series and is an arithmetic series.

The difference between and is that the first series has a value of and so each successive term gets bigger and bigger. The Second series has a value of which is a fraction and so each successive term gets smaller and smaller. This means that we add less and less to the overall sum the more terms we add together. Eventually, terms get so small that we are barely adding anything to the sum.

In the previous activity we discovered that some infinite geometric series (those where there is no limit on the number of terms in the series) do give finite answers. However, we saw that this is only the case if the value of is a fraction.

In general, if an infinite geometric series has a value of , then it will have a finite sum. We can calculate the finite sum of an infinite geometric series by using this formula:

However, we normally write it without the by reversing the order of terms in the denominator.

You will sometimes see infinite geometric series that have a finite sum called **convergent series**, because their sum converges on a single value as gets larger and larger.

If or for a geometric series, the sum is or . We call these **divergent** series. Remember that arithmetic series never converge. They are also divergent.

Watch the video called [Worked example: convergent geometric series](https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-2/v/evaluating-infinite-geometric-series) (03:50) to see an example of how to work with these series.

(<https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-2/v/evaluating-infinite-geometric-series>)

### Activity 4: Infinite Geometric Series

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will introduce you to the sum of infinity geometric series. |
| Stopwatch | Suggested TimeYou will need about 45 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Is this a geometric sequence?
2. Will the series converge?
3. What is the sum to infinity?
4. Convert the recurring decimal to a common fraction. .
5. A ball is dropped from a height of 10m. Each time it bounces, it goes half as high as the previous bounce. What is the total vertical distance that the ball will travel? - <https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-2/v/bouncing-ball-distance>
6. A shrub 110cm is planted in a garden. At the end of the first year, the shrub is 120cm tall. Thereafter the growth of the shrub each year is half of its growth in the previous year. Show that the height of the shrub will never exceed 130cm.
7. For which values of will the series converge and what will its sum be?
8. Find :

#### Guided Reflection

1. We are given the sequence .
2. This is a geometric sequence because there is a common ratio of and a first term of .
3. This geometric series will converge because .
4. We were asked to convert the recurring decimal to a common fraction.

The correct answer is .

Watch the video called [Infinite geometric series word problem: repeating decimal](https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-2/v/repeating-decimal-geometric-series) (08:00) for a full worked solution of this question.

(<https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-2/v/repeating-decimal-geometric-series>)

1. The ball will fall 10m than rise 5m, then fall 5m, then rise 2.5m, then fall 2.5m etc.

We can write this as

We can see that our geometric series is where and . But we must not forget the out front that is not part of the series. If we were to write this in Sigma notation we would write is as

Or

So, the sum of the convergent geometric series is .

But we have the out front. Therefore, the total vertical distance travelled by the ball is m

You can also watch the video called [Infinite geometric series word problem: bouncing ball](https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-2/v/bouncing-ball-distance) (05:10) for a full worked video solution of this question.

(<https://www.khanacademy.org/math/ap-calculus-bc/bc-series-new/bc-10-2/v/bouncing-ball-distance>)

1. The shrub starts out at 110cm. After one year, it grows 10cm to 120cm. Every year thereafter it grows half the previous year’s amount. Therefore, after the second year it grows 5cm, after the third year, 2.5cm etc.

We need to show that it will never grow taller than 130cm, in other words it will never grow more than 20cm.

We can show the amount that the shrub grows each year as follows:

We can re-write this as Therefore, this is a geometric series with and .

Therefore, the shrub will never grow more than 20cm more than its starting height of 110cm – it will never grow taller than 130cm.

1. We know that for a geometric series to converge . In the series .

Therefore,

 i.e. can be any value between and .

1. We are given that

This is an arithmetic series with and . Therefore,

So, we know that

We can see that and .

# Sub-Topic 3: Finance

“Money makes the world go round”, or so the saying goes. This is not true, of course. The earth spins because of residual centrifugal forces experienced by the planet as it formed from the coalescing of space debris.

However, money is really important and much of our life is spent thinking and worrying about it. In fact, you are probably reading this right now in order to eventually improve or increase your ability to make money. There is little we can do without money and not having enough money can significantly constrain our choices and options.

Therefore, being financially literate and having the skills, attitudes and values to work effectively and productively with money is one of the most important things we can learn. This is a Mathematics textbook, however, so we cannot spend too much time on the fundamentals of financial literacy. What we can do, though, is give you the Mathematical skills to work with and understand how loans, investments, interest and bonds, to name just a few, work.

If you are interested in learning more about the principles of good money management, then an excellent book to read is “The Richest Man in Babylon” by George Samuel Clason. A useful place to start is to visit the book’s Wikipedia page at <https://en.wikipedia.org/wiki/The_Richest_Man_in_Babylon>. Here you will find an excellent summary of the most important principles.

Let’s start to get into the Mathematics of finance. The reason we are dealing with finance directly after sequences and series is because so much of the Mathematics of finance is actually sequences and series. Now that we know about sequences and series, what they are and how they work, we can apply this knowledge to understand how finance works. So, let’s get going.

## Unit 1: Simple and Compound Growth and Decay

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Understand the difference between simple and compound rates of growth and depreciation(decay);
2. Calculate simple growth and depreciation (decay);
3. Calculate compound growth and depreciation (decay);
4. Understand the difference between nominal and effective rates;
5. Calculate effective rates; and
6. Work with timelines to simplify complex growth and depreciation situations.

Introduction

This unit is all about simple and compound growth and decay. These words may not mean much to you yet but by the end of this unit you will understand the differences between simple interest and compound interest as well as how the financial value of things grow and shrink or decay.

Simple and compound growth and decay are everywhere and so part of our everyday experience of money and finance that most people know about them without realising. Think about the concept of inflation. Almost everyone knows about inflation, and certainly everyone has experienced inflation. Inflation is when you go to the shop and find that you can buy less for your R10 than a few weeks or months ago.

Inflation is an example of compound growth (or decay). If something that used to cost R2.00 goes up by 10%, it will now cost R2.10. If it keeps going up by 10% every year, it will cost R2.21 the next year, then R2.43 the year after that, R2.67 the year after that and R2.94 the year after that. Each year the cost of the item goes up but, importantly, the amount by which it goes up gets greater and greater as well.

Inflation is an example of compound growth because the price of goods goes up by a greater and greater amount each year. We can also consider it as decay in that the value of your money decreases over time.

But we are getting slightly ahead of ourselves. Let’s take a step back and start at the beginning with interest.

**Interest** is the name we give to the money or fee that someone charges you to lend you money. If the bank lends you money, you pay the money back to the bank along with some extra money that the bank charges you to borrow the money in the first place. If you invest money in the bank, the bank pays you a fee for lending them the money. The fee or the amount of interest is always tied to the amount of money borrowed or lent. Interest is always a percentage of the amount of money borrowed or lent.

### Activity 1: Simple and Compound Interest

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will introduce you to the concepts of simple and compound interest, what they are, and how they differ. |
| Stopwatch | Suggested TimeYou will need about 30 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Imagine you invest R1,000 in a Stokvel. The agreement is that you will earn 5% simple interest on your money each year. Simple interest means that you earn interest only on the money that you initially invest.
2. How much interest will you earn after one year? How much money will you have after one year? Remember that
3. How much interest will you earn after two years? How much money will you have after two years?
4. How much interest will you earn after three years? How much money will you have after three years?
5. Complete the following table showing how much money you have each year for the first five years.

|  |  |  |
| --- | --- | --- |
| **Period** | **Interest earned** | **Total** |
| After 0 years() |  |  |
| After 1 year() |  |  |
| After 2 years() |  |  |
| After 3 years() |  |  |
| After 4 years() |  |  |
| After 5 years() |  |  |

1. What kind of sequence is this?
2. Write the expression for the general term of this sequence. How much money will we have after 5 years (the sixth term in the sequence)?
3. How much money will you have after seven years, ten years and twelve years?
4. Write in terms of the amount invested () and the interest rate () and then write a general expression for the amount of money () you will have after any number of years ().
5. Now imagine that you invested the same amount of money in the Stokvel but this time you earned 5% compound interest per year. Compound interest means that you earn interest on the money you previously earned as interest as well as your initial investment.
6. How much interest will you earn after one year? How much money will you have after one year? Remember that
7. How much interest will you earn after two years? How much money will you have after two years?
8. How much interest will you earn after three years? How much money will you have after three years? How much more interest have you earned than at the simple interest rate?
9. Complete the following table showing how much money you have each year for the first five years.

|  |  |  |
| --- | --- | --- |
| **Period** | **Interest earned** | **Total** |
| After 0 years() |  |  |
| After 1 year() |  |  |
| After 2 years() |  |  |
| After 3 years() |  |  |
| After 4 years() |  |  |
| After 5 years() |  |  |

1. What kind of sequence is this?
2. Write the expression for the general term of this sequence and the expression for the sixth term of the sequence (after 5 years)
3. How much money will you have after seven years, ten years and twelve years? How much more money will you have after twelve years of compound interest that twelve years of simple interest?
4. Write in terms of the interest rate () and then write a general expression for the amount of money () you will have after investing an initial amount () for any number of years ().

#### Guided Reflection

1. We were told that we invested R1,000 in a Stokvel at 5% simple interest per year.
2. We know that we will earn 5% interest per year. In other words, we will earn 5% of our total investment as interest. 5% of R1,000 is . So, we will earn R50 in interest and, therefore, have a total of at the end of year 1.
3. Because we only earn interest on our initial investment ever year, after the second year we will again earn in interest. Therefore, after two years we will have .
4. At the end of the third year, we will again earn 5% of our initial investment i.e. R50. So now our total will be .
5. Here is the completed table.

|  |  |  |
| --- | --- | --- |
| **Period** | **Interest earned** | **Total** |
| After 0 years() |  |  |
| After 1 year() |  |  |
| After 2 years() |  |  |
| After 3 years() |  |  |
| After 4 years() |  |  |
| After 5 years() |  |  |

1. We can see that this is an arithmetic sequence because there is a common difference between each term. Each time we add or R50.
2. We know that and . Therefore, .
3. Seven years will be the 8th term in the sequence:

Ten years will be the 11th term in the sequence:

Twelve years will be the 13th term in the sequence:

1. We can see that our value of . Therefore, we can say that

Based on the expression for the general term of an arithmetic sequence, we get the following:

 but

 but is the original investment i.e.

But we saw in g) above that to work out interest after seven years, we need the 8th term in the sequence. But so we can replace with just (we have already subtracted one).

If we take the common factor of out, we get

1. We were told that we invested R1,000 in a Stokvel at 5% compound interest per year.
2. We know that we will earn 5% interest per year. In other words, we will earn 5% of our total investment as interest after the first year. 5% of R1,000 is . So, we will earn R50 in interest and, therefore, have a total of at the end of year 1.

We can express this total as (we have just taken out the common factor of 1000).

1. Because we are earning compound interest, in year two we don’t just earn interest on the original R1,000. We also earn interest on the R50 interest we earned after year one. Therefore, after the second year we will earn in interest. Therefore, after two years we will have .

We can also express this as

1. At the end of the third year, we will again earn 5% but we will earn this interest on our original investment as well as all the interest we earned in year one and two. In other words, we will earn and will have .

We can also express this as

After three years earning simple interest, we had earned R1,150. Therefore, we have earned R7.63 more in interest after three years by earning compound interest.

1. Here is the completed table.

|  |  |  |
| --- | --- | --- |
| **Period** | **Interest earned** | **Total** |
| After 0 years() |  |  |
| After 1 year() |  |  |
| After 2 years() |  |  |
| After 3 years() |  |  |
| After 4 years() |  |  |
| After 5 years() |  |  |

1. We can see that this is a geometric sequence because there is a common ratio between each term. Each time we are multiplying by .
2. We know that and . Therefore, .
3. Seven years will be the 8th term in the sequence:

Ten years will be the 11th term in the sequence:

Twelve years will be the 13th term in the sequence:

We saw that after twelve years of simple interest, we would have R1,600. However, after twelve years of compound interest we will have R1,795.86 which is R195.86. This is almost 20% of our initial investment more.

1. We can see that our value of . Therefore, we can say that

Based on the expression for the general term of a geometric sequence, we get the following:

 but

 but is the original investment i.e.

But we saw in g) above that to work out interest after seven years, we need the 8th term in the sequence. But so we can replace with just (we have already subtracted one).

In the previous activity we saw how simple and compound interest works and that, because compound interest is calculated on the money invested **as well as the interest earned in previous periods**, one earns for more interest.

The same is true when you are tacking a loan. If the bank charges you compound interest, they will earn more interest i.e. you will *pay* far more interest.

We also derived two extremely important and useful expressions. These were the simple and compound growth formulae.

**Simple growth**:

**Compound growth**:

where

* is called the **principle** – the initial amount deposited or borrowed. You will also see this called the **present value** and denoted as .
* is the **interest rate** per year – you will normally see this quoted as per annum or p.a.
* is the number of **periods** over which the deposit or loan has earned or attracted interest.
* is the **amount** or the total of the principle and interest after periods at interest rate. You will also see this called the **future value** and denoted as .

While these formulae are most often used in financial situations, they work in other situations where there is simple or compound growth, as we shall see.

### Activity 2: Simple and Compound Growth

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you an opportunity to apply the simple and compound growth formulae to different kinds of financial and non-financial problems. |
| Stopwatch | Suggested TimeYou will need about 30 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. You invest R100,000 into a savings account at 13.5% interest per annum.
2. How much will your investment be worth after 10 years if you earn simple interest?
3. How much will your investment be worth after 10 years if you earn compound interest?
4. What simple interest rate would you have to earn for your investment to be worth at least as much after 10 years as your compound interest investment would be worth?
5. In 1968, the population of South Africa was 25 million. The population growth averages 1.8% per year. What will the population likely be in the year 2030, if we assume that growth remains constant? Approximate your answer to the nearest million.
6. You borrow R20,000 from the bank to start a small business. The interest rate you are charged is 9.8% p.a.
7. What is the total amount you will repay the bank if you take the loan over five years?
8. What is the total amount you will repay the bank if you take the loan over ten years?
9. Is it cheaper to take the loan over five or ten years?
10. Reginald invests R10,000 into a savings account. After 10 years, his money has doubled.
11. What annual interest rate would the account have to pay if it was simple interest?
12. What annual interest rate would the account have to pay if it was compound interest?

#### Guided Reflection

1. You invest R100,000 into a savings account at 13.5% per annum. Therefore and .
2. If we earn simple interest, we need to use the simple growth formula.
3. If we earn compound interest, we need to use the compound growth formula.
4. We have to find the simple interest rate that would mean that we earn at least R354,779.58 after 10 years. Therefore , and and we need to solve for .
5. We are told that the population of South Africa was 25 million in 1968 and that it grows by 1.8% per year. This is an example of compound growth because as the population grows, there will be more adults having children (like earning interest on interest). We need to find the expected population in 2030.

We know and . From 1968 to 2030, there will be 62 years. Therefore . We need to find .

Therefore, there will be approximately 76 million people.

1. You borrow R20,000 from the bank to start a small business. The compound interest rate you are charged is 9.8% p.a. Therefore, and .
2. If we take the loan over 5 years, .
3. If we take the loan over 10 years, .
4. Paying the loan back over ten years is much more expensive. If you pay it back over five years you will have to pay in interest. If you pay it back over ten years you will have to pay in interest.
5. We know that Reginald’s original investment of R10,000 grows to become R20,000 in 10 years. Therefore, , and . We need to calculate .
6. For simple interest
7. For compound interest

Sometimes in life, things shrink rather than grow. When we talk about the value of things shrinking in financial terms, we use the word **depreciate** (to lose value or decrease in value). Think about a new car. It is estimated that as soon as you drive a new car off the showroom floor, it loses 20% of its value.

Or think about that old microwave oven you bought for R2,000 five years ago. You would probably struggle to get R200 for it now.

Just like growth, depreciation can be simple or compound in nature. In other words, the rate of depreciation can be based on the starting value only, or it can be based on the previous period’s value (i.e. taking into account all previous devaluation as well). Let’s investigate.

### Activity 3: Simple and Compound Depreciation

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you an opportunity to investigate simple and compound decay or depreciation. |
| Stopwatch | Suggested TimeYou will need about 30 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. A car is worth R280,000 new and depreciates in value by R14,000 every year.
2. What is the rate of depreciation in terms of the car’s starting value?
3. Complete the following table of the car’s value.

|  |  |  |  |
| --- | --- | --- | --- |
| **Year** | **Value at beginning of the year** | **Depreciation amount** | **Value at the end of the year** |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |
| 5 |  |  |  |

1. What kind of sequence does the car’s depreciating value follow?
2. What will the car’s value be at the end of year ten?
3. Write a general expression for the car’s value () at the end of any year in terms of the starting value (), the rate of depreciation () and the number of years ().
4. Plot the values in the table for the car’s value (y-axis) vs the year (x-axis).
5. A company buys a new bakkie worth R500,000. They know that it will lose 10% of its value every year.
6. Complete the following table of the bakkie’s value.

|  |  |  |  |
| --- | --- | --- | --- |
| **Year** | **Value at beginning of the year** | **Depreciation amount** | **Value at the end of the year** |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |
| 5 |  |  |  |

1. What kind of sequence does the bakkie’s depreciating value follow?
2. What will the bakkie’s value be at the end of year ten?
3. Write a general expression for the bakkie’s value () at the end of any year in terms of the starting value (), the rate of depreciation () and the number of years ().
4. Plot the values in the table for the bakkie’s value (y-axis) vs the year (x-axis).

#### Guided Reflection

1. We know that the value of the car new is R280,000 and that it depreciates by R14,000 each year.
2. This means that the rate of depreciation is .
3. Here is the completed table.

|  |  |  |  |
| --- | --- | --- | --- |
| **Year** | **Value at beginning of the year** | **Depreciation amount** | **Value at the end of the year** |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |
| 5 |  |  |  |

1. The car’s value follows an arithmetic sequence where and (or 5% of R280,000).
2. We know that the car’s value at the end of year five is . Therefore, at the end of year ten it will be .
3. In our expression for the value of the car at the end of year five, we know that , and . Therefore, a general expression would be

 (we can take the common factor of out of both terms)

1. Here is the graph of value at the end of the year vs the year. We can see that the values follow a straight line.



1. We know that a bakkie, valued at R500,000 and depreciates at 10% of its value every year.
2. Here is the completed table.

|  |  |  |  |
| --- | --- | --- | --- |
| **Year** | **Value at beginning of the year** | **Depreciation amount** | **Value at the end of the year** |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |
| 5 |  |  |  |

1. We can see that this is a geometric progression where the first term is the bakkie’s new value (R500,000) and the common ratio is .
2. If the bakkie’s value at the end of year five is given by , then its value at the end of year ten will be .
3. In our expression for the value of the bakkie at the end of year five, we know that , and . Therefore, a general expression would be
4. Here is the graph of the bakkie’s values vs year. We can see that these values do not follow a straight line.



Did you notice how similar the expressions for simple and compound depreciation are to those for simple and compound growth?

**Simple growth**:

**Simple depreciation:**

**Compound growth**:

**Compound depreciation:**

We also saw that the values after each period of simple depreciation follow a straight line. This is why simple depreciation is usually called **straight-line** depreciation. After each period, the value of the thing reduces by the same fixed amount.

Compound depreciation, on the other hand, does not follow a straight line. It is usually called **reducing-balance** depreciation. Here the amount by which the thing reduces in value after each period, depends on its value at the end of the previous period. This means that something experiencing reducing balance depreciation, reduces by less and less each period.

### Activity 4: Straight-Line and Reducing Balance Depreciation

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you an opportunity to apply the straight-line and reducing balance depreciation formula in different situations. |
| Stopwatch | Suggested TimeYou will need about 50 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. A second-hand tractor worth R60,000 has a limited useful life of 5 years and depreciates at 20% per annum on a straight-line basis. What will the tractor be worth after five years?
2. The number of cormorants at the Amanzimtoti river mouth is decreasing at a compound rate of 8% p.a. If there are now 10 000 cormorants, how many will there be in 18 years’ time?
3. A woman buys a car for R55,000.
4. What is the value of the car at the end of four years if it depreciates at 4% p.a. on a straight-line basis?
5. What rate of reducing-balance depreciation would result in the car having the same value after the same period of time?
6. A 20 kg watermelon consists of 98% water. If it is left outside in the sun it loses 3% of its water each day. How much does it weigh after a month of 31 days?
7. Richard bought a car 15 years ago and it depreciated by 17% p.a. on a compound depreciation basis. How much did he pay for the car if it is now worth R5,256?
8. Themba buys a car for R330,000 that he knows will depreciate at 13% p.a. on a reducing-balance basis. He wants to buy the same car new in six years’ time. New cars increase in price by 5.5% each year. If he wants to buy the new car in six years’ time with the money from the old car and cash from a savings account, ow much does he need to invest now into the account if it pays 11.5% p.a. compounded yearly.

#### Guided Reflection

1. We are told that the tractor reduces by 20% per year on a straight-line basis. Therefore, its value after five years will be
2. We are told that the reduction of birds is 8% per year on a compound or reducing balance basis. There are 10,000 birds now.

Therefore, there will be 2,229 birds left in 18 years’ time.

1. The present value of the car is R55,000.
2. Straight line depreciation at 4% p.a. for 4 years will mean that the car’s value will be
3. We know that the future value of the car must be R46,200. We also know that starting value and the period of depreciation. So, we need to solve for the rate.
4. We know that the watermelon weights 20 kg watermelon and that 98% of this weight is water. This means that it contains kg of water. Loosing 3% of its water each day is a compound reduction. So, after 31 days, it will have

kg of water.

This means that it will have lost kg of water and that it will be kg in weight.

1. We know that the current value of the car is R5,256 and that it has depreciated by 17% p.a. compound depreciation over 15 years. We need to work out what the original value of the car was.
2. Let’s first work out what the value of Themba’s car will be in six years’ time. It is currently worth R330,000 and will depreciate at 13% p.a. on a reducing balance basis.

Now, let’s work out what the value of the new car will be in six years’ time.

This means that Themba’s investment will need to have a value of .

We know that the account pays 11.5% compound interest and that its future value must be R299,134.91.

This means that Themba needs to invest R155,674.64 now in order to buy the new car for cash.

So far in this unit, we have always assumed that a compound interest rate (or compound rate of depreciation), is compounded only once a year. In other words, we have assumed that the interest is calculated only once a year.

In reality, however, most compound interest is calculated and paid on a monthly basis. If you have a savings account, you probably earn interest on your money every month. How do we cater for these situations? Thankfully, the answer is simple.

Let’s say that a savings account pays 6% interest p.a. but that it is compounded monthly. What this means is that every month, the money in the account earns interest and that this interest is calculated and paid 12 times per year. In two years, the interest will be paid and calculate times.

We call the quoted interest rate of 6% p.a. compounded monthly the **nominal rate**. The **effective annual rate** is the interest rate compounded annually that would be needed to earn the same amount of interest in one year.

An example will help to illustrate all this.

Let’s say you invest R2,000 for three years at a nominal rate of 7% p.a. compounded monthly. In order to work out the total amount of money you will have after three years, we still use the same compound growth formula but with two important changes.

1. Because the compounding happens monthly (12 times a year), we have to **divide the interest rate by 12**.
2. Because there will be 12 compounding events per year, we have to **multiply the number of years by 12**.

 where , and

Let’s investigate nominal and effective rates some more.

### Activity 5: Nominal and Effective Rates

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you an opportunity to explore the differences between nominal and effective rates in more detail. |
| Stopwatch | Suggested TimeYou will need about 50 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. An investment of R10,000 is made into an account with a nominal interest rate of 9.8% p.a. compounded monthly.
2. How much is the investment worth after one year?
3. How much is the investment worth after ten years?
4. What interest rate compounded annually would result in the same value of the investment after one year?
5. You have R15,000 to invest for five years. You have a choice of three accounts – one with an interest rate of 6.5% p.a. compounded monthly, one with an interest rate of 7.0% p.a. and one with an interest rate of 6.75% p.a. compounded quarterly (quarterly means once a quarter or once every three months). Which account should you choose to maximise your returns?
6. Convert the following nominal interest rates:
7. 9% p.a. compounded every six months into an effective annual rate (to three decimal places).
8. 5% p.a. compounded daily into an effective annual rate (to three decimal places).
9. 5% p.a. compounded monthly into an effective quarterly rate (to three decimal places).

#### Guided Reflection

1. We are told that R10,000 is invested at a nominal rate of 9.8% p.a. compounded monthly. This means that the interest rate applied every month is .
2. After one year, there will have been 12 compounding periods. This means that .

 where , and

1. After 10 years, the number of compounding periods will be . Therefore,

 where , and

1. We know that the investment is worth R11,025.28 after one year. If (only one compounding event) we need to find what needs to be to result in this same future value.

 where , and

This means that a nominal rate of 9.8% p.a. compounded monthly is the same as an effective rate of 10.25% p.a. compounded annually.

1. We need to work out which interest rate will make our R15,000 investment grow the most in five years.
2. 6.5% p.a. compounded monthly:

 where , and

1. 7.0% p.a. (if no compounding periods are specified, we assume these are the same as the quoted rate i.e. per annum)

 where , and

1. 6.75% p.a. compounded quarterly

 where , and

The best return is from the account paying 6.5% p.a. compounded monthly, even though the other nominal interest rates were higher.

1. We need to convert nominal interest rates into effective interest rates. Let the nominal rate be and the effective rate be . We know that the effective rate is the rate that, with just one compounding, would produce the same result as the nominal rate.

Therefore, in all cases and where is the number of nominal compounding periods in one effective compounding period.

1. 9% p.a. compounded every six months into an effective annual rate

 where and (there are 2 six-month periods in a year)

 (divide both sides by )

1. 5% p.a. compounded daily into an effective annual rate

 where and (there are 365 days in a year)

 (divide both sides by )

1. 5% p.a. compounded monthly into an effective quarterly rate.

 where and (there are 3 months in a quarter of a year)

 (divide both sides by )

Sometimes, growth and depreciation situations can get quite complicated and it helps to use a timeline to show what happens and when it happens. Let’s take a look.

### Activity 6: Timelines

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will introduce you to timelines as a tool for helping you to think through more complicated growth and decay situations. |
| Stopwatch | Suggested TimeYou will need about 45 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. R150,000 is deposited in an investment account for a period of 6 years at an interest rate of 12% p.a. compounded half-yearly for the first 4 years and then 8.5% p.a. compounded monthly for the rest of the period. A deposit of R8,000 is made into the account after the first year and then another deposit of R2,000 is made five years after the initial investment. Finally, a withdrawal of R2,500 is made five and a half years after the initial investment. Below is a timeline showing the six years of the investment with being the start of the investment period.



1. Make a copy of this timeline and add in the points at which the initial R15,000 was invested, when the two additional deposits were made and when the withdrawal was made.
2. Now, by using horizontal arrows, indicate on the timeline when the interest rate was 12% p.a. compounded half-yearly and when it was 8.5% p.a. compounded yearly.
3. Calculate the total growth of the initial deposit over the first four years.
4. Calculate the total growth of the initial deposit over the final two years.
5. Write an expression that combines the growth of the initial deposit over both periods.
6. Calculate the growth of the R8,000 deposit over its full investment term.
7. Calculate the growth of the second R2,000 deposit over its full investment term.
8. Calculate the growth that the R2,500 withdrawal would have experienced if it had stayed invested to the end of the six-year period
9. Calculate the overall value of the investment at the end of the 6 years.
10. Sindisiwe wants to buy a motorcycle. The cost of the motorcycle is R55,000. In 1998, Sindisiwe opened an account at Sutherland Bank with R16,000. Then in 2003 she added R2,000 more into the account. In 2007, Sindisiwe made another change: she took R3,500 from the account. If the account pays 6% p.a. compounded half-yearly, will Sindisiwe have enough money in the account at the end of 2012 to buy the motorcycle?
11. R75,000 is invested in an account which offers interest at 11% p.a. compounded monthly for the first 24 months. Then the interest rate changes to 7.7% p.a. compounded half-yearly. If R9,000 is withdrawn from the account after one year and then a deposit of R3,000 is made three years after the initial investment, how much will be in the account at the end of 6 years?

#### Guided Reflection

1. This was a complicated situation, so we need to work through it step by step.
2. We have added in the initial investment of R15,000, the two additional deposits of R8,000 at the end of year one and R2,000 at the end of year five, as well as the withdrawal five and a half years after the initial investment.



1. Here we have added in the interest rates at different points in the six-year period.



1. For the first four years, the nominal interest rate is 12% p.a. compounded half-yearly. That means that the growth of the initial deposit will be

 where , and

1. For the last two years, the nominal interest rate is 8.5% p.a. compounded monthly. The starting amount will be the result of the first four years’ growth. That means that the growth of the initial deposit for this period will be

 where , and

1. From d) we know that the final value of the initial deposit is

But from c) we know that .

Therefore, we can combine these to get that the final value of the initial deposit is given by

1. The R8,000 was deposited a year after the initial deposit. Therefore, it grew at the initial 12% p.a. compounded half-yearly rate for three years and the final 8.5% p.a. compounded yearly rate for two years. We can just edit our expression from e).
2. The R2,000 deposit was made at . Therefore, it only experienced the final 8.5% p.a. compounded yearly rate for one year. Therefore,
3. The R2,500 would have experienced six month’s growth at 8.5% p.a. compounded monthly. This is given by

 where , and

1. To calculate the value of the overall investment, we basically just need to add these separate amounts up.
2. TBC
3. TBC

## Unit 2: Annuities, Loans and Investments

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Understand the difference between simple and compound rates of growth and depreciation(decay);
2. Calculate simple growth and depreciation (decay);
3. Calculate compound growth and depreciation (decay);
4. Understand the difference between nominal and effective rates;
5. Calculate effective rates; and
6. Work with timelines to simplify complex growth and depreciation situations.

Introduction

In many real-life situations, we do not make a single investment into a savings account. Rather, we invest a smaller amount (usually the same amount every month) into a savings or investment account. Also, when we take a loan, or buy something on credit, we almost always pay it back by making monthly repayments of the same amount rather than a single payment after a specific amount of time.

While the simple or compound growth formula can give us the final value of a single deposit investment, it is not easily to use this on its own to tell us the value of an investment that we regularly contribute to.

In the same way, the simple or compound growth formula cannot easily be used to tell us about the size of the monthly repayments we need to make to pay down a debt in a specific amount of time.

For these situations where there are regular payments, we need to expand or modify the simple and compound formulae. Having these in our toolbox, will help us to critically analyse real investment and loan options and to make the best, most-informed financial decisions for us and our families.

As usual, let’s start with an investigation, this time on the value of saving regularly, even if it just a small amount.

### Activity 1: Save Regularly

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will introduce you to the power of saving, even small amounts, regularly. |
| Stopwatch | Suggested TimeYou will need about 45 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Imagine you save R200 every month into a savings account that pays 8.25% p.a. compounded monthly for three years. Here is a timeline of your savings.



1. Why are 36 intervals shown?
2. Why is there no payment for ?
3. What is the value of your savings immediately after the first deposit?
4. What is the value of your savings immediately after the second deposit?
5. What is the value of your savings immediately after the third deposit?
6. What is the value of your savings immediately after the 36th deposit?
7. What kind of series did you create in f)?
8. Write an expression for the sum of this series and hence calculate the value of your investment after 36th deposit.
9. What is the value of your savings after the full three-year period?
10. Write a general expression for the value of this kind of regular saving () in terms of the amount saved (), the interest rate () and the number of payments made ().
11. How much interest was earned in total?
12. What lump sum (single deposit) would have to be made for the same period at the same interest rate to end up with the same final value.

#### Guided Reflection

1. We save R200 every month for three years and earn 8.25% p.a. compounded monthly.
2. The term of our investment is three years, and, in this time, we will make 36 equal deposits, hence we need 36 intervals to account for the interest earned by these deposits. If the first deposit is made at it will earn its first interest payment at , one month later. The final deposit at will earn its interest at , again, one month later.
3. There is no payment show at because, this is the end of the three-year term. Any deposit made here, would not earn any interest.
4. Immediately after the first deposit, the value of the investment is .
5. Immediately after the second deposit, the first deposit would have attracted one month’s interest. Therefore, the value of the investment would be
6. Immediately after the third deposit the second deposit would have attracted one month’s interest and the first deposit would have attracted two month’s interest. Therefore, the value of the investment would be
7. Immediately after the 36th deposit the value of the investment would be
8. This is a finite geometric series where , and (there are 36 terms in the series).
9. We know that the sum of a finite geometric series is given by . Therefore, the sum of our series is given by
10. This total will then attract one more month’s interest.
11. We saw that, for 36 payments

But , , and . Therefore, in general,

1. We made 36 payments of R200. This means that we made in deposits. However, the total value of the savings was R7,954.35. Therefore, we earned in interest.
2. We want to have a final value , with and .

If you do not have R6,169.50 to save now, you can still have the same amount of money in three years’ time if you just saved R200 a month.

Remember that when we worked with compound growth and depreciation, we were actually working with geometric sequences. Well it turns out, that if we have regular payments, like in Activity 1, rather than a single payment, we have to work with geometric series instead.

We call any series of regular payments, whether they are savings deposits or loan re-payments, an **annuity**.

In the example in Activity 1, our general expression for an annuity gives us the value **immediately after the last** (nth) payment. Because we are calculating the future value of the series of payments, we call this a **future value annuity**.

**Future value annuity:**

Where are the regular deposits or payments, is the interest rate and is the number of payments made.

If we let the money grow further without any additional payments, we need to calculate this growth separately using the normal compound growth formula.

Let’s get some more practice working with future value annuities of different kinds.

### Activity 2: Future Value Annuities

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will introduce you to the power of saving, even small amounts, regularly. |
| Stopwatch | Suggested TimeYou will need about 60 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. A teacher makes regular payments, at the beginning of each month, of R450 into a fund offering 6% p.a. compounded monthly for 20 years.
2. How much will be in the account at the end of 20 years?
3. How much will be in the account if she stops making payments after 20 years and lets the money grow at 7% p.a. compounded quarterly?
4. Shelly decides to start saving money for her son’s future. At the end of each month she deposits R500 into an account at Durban Trust Bank, which earns an interest rate of 5.96% per annum compounded quarterly.
5. Determine the balance of Shelly’s account after 35 years.
6. How much money did Shelly deposit into her account over the 35-year period?
7. Calculate how much interest she earned over the 35-year period.
8. Gerald wants to buy a new guitar worth R7400 in a year’s time. How much must he deposit at the end of each month into his savings account, which earns an interest rate of 9;5% p.a. compounded monthly?
9. Lerato plans to buy a car in five and a half years’ time. She has saved R30,000 in a separate investment account which earns 13% per annum compounded quarterly. If she doesn’t want to spend more than R160,000 on a vehicle and her savings account earns an interest rate of 11% p.a. compounded monthly, how much must she deposit into her savings account each month?
10. A delivery company buys a truck for R350,000. The value of the truck depreciates by 17% per annum on a reducing-balance basis. The company plans to replace this truck in seven years’ time, and they expect the price of a new truck to increase annually by 9%.
11. Calculate the value (usually called the book value) of the truck in seven years’ time.
12. If the company uses the sale of the existing truck to help fund the purchase of the new truck, how much extra money will the company need to buy the new truck?
13. What monthly deposits must the company start to make now into an account bearing 12.25% p.a. compounded monthly to be able to buy the new truck?

#### Guided Reflection

1. We know that the monthly deposit is R450 and that the interest rate in 6% p.a. compounded monthly.
2. Here is a timeline showing the situation for the 20 years.



First, we need to work out how much money is in the account immediately after the 240th payment.

 where , and

But we need to find out the value after the full 20 years. In other words, this amount of money remains in the account for one extra month.

 where , and

1. Now the money she has accumulated is going to grow for another 10 years at 7% p.a. compounded quarterly.

 where , and

1. Shelly deposits R500 per month and the interest rate is 5.96% p.a. compounded quarterly.

Here is a timeline illustrating this situation.



Because she deposits at the end of the month, her last deposit will be at . This means we do not have to account for a final period of interest after the last payment.

Also, interest is calculated once a quarter which means that she makes three R500 payments before interest is calculated each time. This is the same as making a single R1,500 deposit every three months. This means that in our future value annuity formula .

1. where , and
2. Shelly made monthly deposits of R500. Therefore, she deposited .
3. Shelly received in interest.
4. Because Gerald makes payments at the end of the month, we do not need to account for an extra month of interest. We can just use the future value annuity formula. But this time, we need to solve for . The regular payment.

 where , and

1. Lerato needs to have R160,000 in order to buy her new car. The first thing we need to do is work out how much of this money will come from her separate investment account.

 where , and

This means that her savings account needs to have in it at eth end of five and a half years.

 where , and

1. We know that the current truck is worth R350,000 now and that the company wants to by a similar truck in 7 years’ time.
2. To calculate the value of the truck in 7 years’ time, we need to use our compound or reducing-balance depreciation formula.

 where , and

This means that R94,976.18 of the money needed to buy the new truck will come from the sale of the current truck in seven years’ time.

1. We need to work out what the cost of a new truck in seven years’ time will be based on the fact that trucks get more expensive by 9% every year.

 where , and

In seven years’ time, a new truck will cost R639,813.69. This means that the company will need an additional to buy a new truck.

1. We need to work out the monthly deposits that are necessary for a future annuity to be worth R544,837.51.

 where , and

The company needs to deposit R4,128.98 every month to be able to afford the new truck in seven years’ time.

The final question in the previous activity illustrates what a **sinking fund** is. Typically, companies will set up a sinking fund in order to save for the replacement of vehicles, equipment or machinery. But, of course, individuals can save like this as well, especially for expensive items like cars, education or home improvements.

A sinking fund is really nothing more than an account into which regular annuity payments are made so that a specific amount of money has accumulated in the account after a specific amount of time. Notice that sinking funds are entirely based on future value annuities because we are aiming at having a specific future value.

Let’s have a look at a few more sinking fund examples.

### Activity 3: Sinking Funds

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you more experience working with future annuities in the form of sinking funds. |
| Stopwatch | Suggested TimeYou will need about 45 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Atlantic Transport Company buys a van for R265,000. The value of the van depreciates on a reducing-balance basis at 17% per annum. The company plans to replace this van in five years’ time, and they expect the price of a new van to increase annually by 12%.
2. Calculate the book value of the van in five years’ time.
3. Determine the amount of money needed in the sinking fund for the company to be able to afford a new van in five years’ time.
4. Calculate the required monthly deposits if the sinking fund earns an interest rate of 11% p.a. compounded monthly.
5. Geoff wants to build an extra room in his house for his parents when they retire in eight years. He estimates that he will need R235,000 to complete the alterations. If he can save R1,245 a month into an account earning 12% p.a. compounded monthly, when does he need to start saving?

#### Guided Reflection

1. This is a very typical sinking fund problem where you need to first work out the future value of a depreciating asset so that you can work out how much additional money will be required to buy a new asset.
2. The van is currently worth R265,000 but depreciates at 17% p.a. (reducing-balance).

 where , and

The van will be worth R104,384.58 in five years.

1. New vans increase in price by 12% each year. We need to work out the cost of a new van in five years.

 where , and

A new van will cost R467,020.55 in five years but we will have R104,384.58 available from the sale of the existing van. Therefore, .

This means that the sinking fund must be worth R362,635.97 in five years.

1. Now we need to work out what monthly deposits will be required to create a sinking fund of the necessary value.

 where , and

This means the company has to deposit R4,560.42 every month for five years to be able to afford a new van.

1. We know that Geoff needs to have R235,000 available in ten years’ time. We also know that he can save R1,245 a month. What we need to find out is how many such payments he will need to make, and therefore, when he needs to start saving.

 where , and

To solve for we need to use logs. Remember we learnt how to solve log equations earlier on in this topic. We know that if then . We also know that . Therefore,

This answer for is in terms of months which means that Geoff will need to make at least 107 monthly payments to have enough money. 107 monthly payments mean that he needs to save for years or 8 years and 11 months. He needs to start saving one year and one month from now.

Future value annuities, as we have seen, are all about what the future value of a series of regular payments will be. This is great for sinking funds and working out the future value of savings and investment plans. But what if you need money right now?

For the vast majority of people throughout the world, there is simply not enough time available for them to save for the really big and important things, like a home. Everyone needs a home now. For most people, this means taking a loan from a bank so that they can buy a home now.

Many businesses, especially small, start-up businesses also need money now to buy the things that will make the business work so that they have the ability to make money in the future. They also need to take a loan from a bank or even a micro-finance organisation.

Loans for things like homes or activities that help make people economically productive are extremely beneficial for almost everyone. Loans for things like the latest flatscreen TV or fancy BMW, well, those are probably less of a good idea. There is good debt and bad debt.

Good debt is debt for essentials or that helps one to become more economically productive. Bad debt is debt taken to fund non-essential, wasteful or non-productive consumption.

Enough talking about loans, let’s see how they actually work.

Most of the time, loans are structured to be paid back by means of annuity payments, a set number of regular payments of a set amount. These payments are determined so that they pay off the actual loan (often called the capital) and the interest the lender charges for giving the loan in the first place.

Because loans give you the money you need now, they are an example of **present value annuities**. It is the present value that is defined.

### Activity 4: Present Value Annuities

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will introduce you to present value annuities and to derive the present value annuity formula. |
| Stopwatch | Suggested TimeYou will need about 50 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. A university student is awarded an NSFAS bursary. She knows that her monthly living expenses for the year are going to be R2,700. She wants to deposit a portion of her bursary money into a savings account that offers 9.8% p.a. compounded monthly so that she can withdraw R2,700 each month for the year. How much money does she need to deposit by the beginning of December so that she can make her first withdrawal at eth beginning of January?
2. Work out how much she would have to deposit to make only a single withdrawal in January.
3. Work out how much she would need to deposit to make two withdrawals (in January and February).
4. Work out how much she would need to deposit to make three withdrawals (in January, February and March).
5. Complete the following table.
6. What kind of series is formed and what are its characteristics?
7. Use this series to work out how much she must deposit to make the full twelve withdrawals.
8. If for , write an expression to work out the amount that must be deposited now () to allow equal withdrawals of if the interest rate is . Hint: It is very similar to the future value annuity formula.
9. Prineshni wants to buy a house. She knows that she can make monthly payments of R9,700 and that the bank will change her 12.5% p.a. compounded monthly for a 20-year bond. What is the maximum she can spend on her house?
10. Work out what the future value of her regular payments over 20 years at 12.5% p.a. compounded monthly would be.
11. If the loan she can take now is , write an expression for what this amount plus interest over the 20 years would be.
12. If we know that plus interest, write another expression for .
13. Calculate , the loan Prineshni can take now.
14. Using the expression from c), write a general expression for the size of the loan that can be taken if there are equal payments of and the interest rate is .

#### Guided Reflection

1. Because we need to work out what the present value of the deposit must be to allow regular monthly withdrawals, this is an example of a present value annuity.
2. If she wanted to make just one withdrawal, her deposit would grow for a single month

 where , and

1. We already know how much must be deposited for the first withdrawal. The deposit for the second withdrawal is going to be smaller because it will earn more interest. Let’s work out what this deposit would have to be.

 where , and

Therefore, the deposit required for two withdrawals will be

1. We already know how much must be deposited for the first two withdrawals. The deposit for the third withdrawal is going to be smaller because it will earn more interest. Let’s work out what this deposit would have to be.

 where , and

Therefore, the deposit required for two withdrawals will be

1. This is a geometric series where , and .
2. For the full twelve withdrawals, we need to work out the sum of this series to twelve terms.
3. We know that and that .

Let’s take hat is being multiplied by from the numerator into the denominator.

1. In this case, no monthly withdrawals are being made. Instead, a present amount is being accessed now that will be paid back by means of regular payments.
2. Let’s assume that Prineshni was actually saving for her house through regular deposits into an account. We want to find out what this investment would be worth in 20 years.

 where , and

By saving like this, Prineshni could save R10,267,387.86.

1. If the loan she can take now is , then this amount, plus interest in 20 years would be
2. We know that , therefore, we can replace A in the expression above with

Prineshni is able to take a R853,767.61 loan from the bank now.

1. From c) we know that . We can generalise this as follows:

This is exactly the same expression we got in question 1f)

In the previous activity we saw two different situations involving present value annuities:

1. Where a single deposit is made now to enable future regular withdrawals of the same amount; and
2. Where a loan amount is paid out now and then repaid by means of regular payments of the same amount.

We also saw that the expression for both situations was exactly the same.

**Present value annuity:**

Where are the regular withdrawals taken or repayments made, is the interest rate and is the number of withdrawals taken or repayments made.

Compare this expression with the future value annuity one to see how similar they are.

**Future value annuity:**

You should learn both these expressions, so you don’t get confused.

As we said earlier, loans and debt are not inherently bad. It all depends on what you take the loan for. Taking loans for essentials or things that will make you more economically productive can be very good. Taking loans to fund non-essential consumption can be very bad and lead to financial ruin. Let’s investigate further.

### Activity 5: Loans

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you an opportunity to evaluate the cost of taking loans so that you can decide if it is worth it or not.  |
| Stopwatch | Suggested TimeYou will need about 75 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. A house bond is paid back (or amortised) by 240 monthly payments of R3,000. Determine the amount of the bond if the interest is 12.5% p.a. compounded monthly.
2. A retired couple have a total pension of R1.5 million. They decide to buy an annuity that will give them regular monthly payments to live on. How much can they withdraw each month if the annuity pays 11.2% p.a. compounded monthly and they expect to need to make withdrawals for 25 years?
3. The international Lotto offers a prize worth R5 million paid out as R1 million immediately and R160,000 each year for 25 years starting in a year’s time. The Lotto company can buy an annuity giving 10% p.a. compounded daily.
4. What is the effective rate of the annuity?
5. How much money does the Lotto company need to pay out the R1 million and invest in the annuity on the day of the draw?
6. You want to buy a new TV. The TV you like costs R15,000. You can either save up for it or take a loan to buy it.
7. How much do you need to save each month for 2 years in a savings account offering 12.8% p.a. compounded monthly, if the cost of the TV increases by 5% per year.
8. If you buy the TV on credit, you need to pay a 10% deposit and then equal monthly payments for 2 years at an interest rate of 16% p.a. compounded monthly. How much will you end up paying for the TV altogether?
9. Which option do you think is better?
10. Hristo wants to buy a small wine farm worth R 8 500 000. He plans to sell his current home for R3,400,000 which he will use as a deposit for the purchase of the farm. He secures a loan with HBP Bank with a repayment period of 10 years and an interest rate of 9;5% compounded monthly.
11. Calculate his monthly repayments.
12. Determine how much interest Hristo will have paid on his loan by the end of the 10 years.
13. Dullstroom Bank offers personal loans at an interest rate of 15.63% p.a. compounded twice a year. Lubabale borrows R3,000 and must pay R334.93 every six months until the loan is fully repaid.
14. How long will it take Lubabale to repay the loan?
15. How much interest will Lubabale pay for this loan?
16. David and Julie take out a home loan of R2.6 million with an interest rate of 10 % per annum compounded monthly.
17. Calculate the monthly repayments for a repayment period of 30 years.
18. Calculate the interest paid on the loan at the end of the 30-year period.
19. Determine the monthly repayments for a repayment period of 20 years.
20. Determine the interest paid on the loan at the end of the 20-year period.
21. How much money could they save by paying the loan off faster?

#### Guided Reflection

1. This is a simple question to start off with. Because it deals with paying back a loan, we know it is a present value annuity problem.

 where , and

The bond is for R264,051.84.

1. We know that the present value of the annuity will be R1.5 million and what the interest rate and number of withdrawals will be. We need to calculate what these withdrawals will be i.e. the value of .

 where , and

They will be able to draw R3,729.77 every month for 25 years.

1. This annuity problem is a little more complicated.
2. We are told that the nominal interest rate is 10% p.a. compounded daily. We know from our work on nominal and effective rates that the effective rate can be considered that rate that would produce the same growth or depreciation after one year.

Therefore, in all cases and where is the effective rate and is the nominal rate. We can always dive through by to get

 where (there are 365 days in one year)

The effective rate is 10.52% p.a.

1. We know that the annuity needs to pay R160,000 every year for 25 years and that the effective annual rate is 10.52%.

 where , and

The present value of the annuity (the amount that must be deposited now) is R1,396,150.55. Therefore, the Lotto company will need R2,396,150.55 on the draw day.

1. Buying non-essentials like TVs on credit (with a loan) is not always a very good idea.
2. If you buy the TV in 2 years’ time, it will cost more. We are told that its price goes up by 5% each year.

 where , and

In two years, the TV will cost R16,537.50.

This means the future value of our savings (our future value annuity) must be R16,537.50.

 where , and

You would need to save R608.27 in order to buy the TV in two years.

1. The 10% deposit means that you have to pay R1,500 upfront and take a loan for the balance which would be R13,500. So, the present value of your loan would be R13,500.

We first need to see what our monthly payments would be.

 where , and

You would have to pay R661 each month for two years.

This means that you will pay for the TV altogether.

1. Buying the TV later for cash after you have saved means that you will pay less overall for the TV. You will pay R16,537.50 as opposed to R17,364.00. This is a difference of R826.50. There is no one correct answer. Buying the TV now means you will pay more overall, but it also means that you have the TV now. It all depends what is more important to you – R826.50 or the TV now. In many situations, there is not likely to be one clearly correct choice. It will often come down to what you value but at least you will be able to make these decisions with all the facts rather than just on a feeling.
2. This is a reasonably straight-forward present value annuity problem as it involves the taking out and repayment of a loan.
3. Hristo will not need a loan for the full amount of the farm because he will have money from the sale of his current house. Therefore, he will need .

 where , and

He will have to pay R65,992.75 each month for ten years to repay this loan.

1. The total amount he will pay to the bank will be .

But the loan was only for R5.1 million. Therefore, he will have to pay in interest.

1. The interest rate offered is 15.63% p.a. compounded twice a year.
2. We need to find , the number of payments needed to pay back the loan in order to work out how long it will take Lubabale to pay it back.

 where , and

This means that Lubabale will need to make 16 payments. But these payments are made twice a year. Therefore, it will take her eight years to pay back the loan.

1. If Lubabale makes 16 payments of R334.93, she will end up paying a total of . But the loan amount was R3000. Therefore, she will pay R2,358.88 in interest.
2. This question will give us some insight into the value of paying down debt as quickly as possible.
3. Over 30 years:

 where , and

Therefore, monthly repayments of R22,816.86 will be necessary.

1. The total amount paid will be . But the loan was only far R2.5 million. This means that R5,614,069.60 will be paid in interest.
2. Over 20 years:

 where , and

Therefore, monthly repayments of R25,090.56 will be necessary.

1. The total amount paid will be . But the loan was only far R2.5 million. This means that R2,192,335.20 will be paid in interest.
2. By paying the loan off over 20 years instead of 30, they will save the difference in the interest payments over 20 vs 30 years i.e. they will save R2,192,335.20.

The result of the final question in the previous activity is quite astonishing. By paying R2,273.70 more per month on their home loan and, thereby paying it off in 20 years instead of 30 years, David and Julie would end up saving more than R2 million.

This is true no matter the size of the loan. The quicker you repay a loan, the less the loan will cost you in terms of interest. If you have any debt and a little bit of extra money, it is almost always far better to pay off the debt quicker by paying extra each month, then putting the extra money into a savings account.

In case, case, however, you now have all the skills you need to do the calculations yourself and make the best choice for you.

Let’s end this section by taking a look at analysing some more financial situations you may come across.

### Activity 6: Financial Analysis

|  |  |
| --- | --- |
| Bullseye | PurposeThis activity will give you an opportunity to evaluate the cost of taking loans so that you can decide if it is worth it or not.  |
| Stopwatch | Suggested TimeYou will need about 90 minutes. |
| Pencil | What You Will Need* A pen or pencil
* Some blank paper or a notebook
 |

#### Tasks

1. Everything Maths Gr12 (pg127 WE12)
2. Everything Maths Gr12 (pg129 WE13)
3. Everything Maths Gr12 (pg131 Ex3-5.2)
4. Everything Maths Gr12 (pg134 Ex3-6.4)
5. Everything Maths Gr12 (pg134 Ex3-6.6a)
6. Everything Maths Gr12 (pg134 Ex3-6.8)
7. Everything Maths Gr12 (pg134 Ex3-6.9)
8. Everything Maths Gr12 (pg134 Ex3-6.10)