# Mathematics

Topic 2: Functions, Graphs and Algebra

NASCA Mathematics

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# Sub-Topic 3: Functions

Functions are one of the primary reasons Maths is so important and useful. Functions let us describe and explore the relationships between different things. They let us design and build real things like buildings, planes, computers and cell phones. They help us predict changes in populations and the economy. They even help to fight diseases like cancer and HIV.

In fact, everything we have done so far in Topic 2 is really just so that we can understand how functions work and use them to do useful things.

So, the R1,000,000 question is “what are functions?” Let’s find out.

## Unit 1: What Are Functions?

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Define what a function is;
2. Explain the difference between a function and a relation; and
3. Represent functions in different ways, including function notation.

#### Introduction

The best way to start to answer the question “what are functions?”, is to do some investigating.

### Activity 1: Discovering Functions

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to discover what functions are. |
| Stopwatch | Suggested Time You will need about 60 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Bottle tops and jar lids * Ruler * String * Calculator * Set square * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. Collect a few bottle tops and jar lids of different sizes (basically circles of different sizes). Now, carefully use your ruler to measure the diameter of each top or lid (that is the straight line distance from one side to the other through the centre) and use the string to measure the circumference (the distance around each top or lid) by tightly wrapping the string around the lid and then measuring the length of string needed to go once all the way around. NOTE: If you cannot get real lids, then just draw a few circles of different sizes on a piece of paper using a compass.

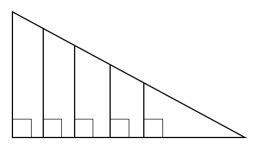
Record your measurements for each top or lid in a table like this one:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Lid 1** | **Lid 2** | **Lid 3** | **Lid 4** |
| Diameter |  |  |  |  |
| Circumference |  |  |  |  |
|  |  |  |  |  |

Now, in the last row, calculate the circumference divided by the diameter for each lid to 2 decimal places

* 1. What do you notice about your answers?
  2. Do you recognise this number? What number is it?
  3. If you knew the diameter of a lid, how could you predict its circumference? Why?
  4. If the diameter of a certain lid is 12cm, what is its circumference be?
  5. If the circumference of a certain lid is 15.24cm what is its diameter?
  6. If you know the diameter is there any chance that there is more than one corresponding circumference?

1. Use a set square (or anything else that has a 90° or right angle) to draw a series of at least 4 right-angled triangles that all have the same angles but are different sizes. Your triangles might look like these.



Pick one of the two non-right angles as your reference angle and measure the length of the side next to this angle (the adjacent side) as well as the length of the hypotenuse (the side opposite the right angle) for each triangle. Enter your measurements into a table like this one.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Triangle 1** | **Triangle 2** | **Triangle 3** | **Triangle 4** |
| Length of side next to the angle (adjacent) |  |  |  |  |
| Length of hypotenuse |  |  |  |  |
|  |  |  |  |  |

Now, in the last row, calculate the length of the adjacent side divided by the length of the hypotenuse.

* 1. What do you notice about your answers?
  2. If you knew the length of the adjacent side, how could you predict its circumference?
  3. If the adjacent side of a triangle with your angles is 45km, what is the length of its hypotenuse?
  4. If the hypotenuse of a triangle with your angles is 13m, what is the length of its adjacent side?
  5. If you know the length of the adjacent side, is there any chance there is more than one corresponding length of hypotenuse?

1. Visit <https://www.thecalculatorsite.com/conversions/area/hectares-to-acres.php>. It has a calculator that converts between hectares and acres. Draw a table similar to this one.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Hectares** | 1 |  | 5 |  |
| **Acres** |  | 5 |  | 12 |

* 1. Use the calculator to complete the table.
  2. Write an equation that shows the relationship between hectares and acres in the form of hectares = ….
  3. Use this equation to work out what 150 hectares are in acres.
  4. Can you ever get more than one answer when converting between hectares and acres?
  5. Plot each of the points in the table above on a Cartesian Plane. Let acres be and hectares by . What kind of line do all the points seem to fall on? If you joined these points with a solid line, how could you use this line to work out how many hectares 10 acres is? How many hectares is this?

1. In South Africa, electricity costs 170c per kWh (kilowatt hour) with a basic delivery charge of R485 per month.
   1. How much will a household pay if they use 50kWh in a month?
   2. How much will a household pay if they use 75kWh in a month?
   3. How much will a household pay if they don’t use any electricity in a month?
   4. Write an equation that describes the relationship between the amount of electricity a household uses and the amount they have to pay.
   5. How much electricity can a household use in a month if they can only spend R600 on electricity?
   6. Plot the relationship between cost and kWh on a Cartesian Plane. Let kWh be . You can use any points you like.
   7. Use your graph to work out how many kWh a household would use if their bill was R1,000.
   8. Can you think of any other relationships between variables that are similar to this one?
2. In South Africa, we measure temperature in degrees Celsius (°C). In the United States, temperature is measured in degrees Farenheit (°F). Since both are measures of temperature, we need a way to convert from one to the other.

Here is a table that shows the same temperatures in °C and °F.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **°C** | -20 | -10 | 0 | 10 | 20 | 30 | 40 |
| **°F** | -4 | 14 | 32 | 50 | 68 | 86 | 104 |

* 1. If the equation that converts °C into °F is of the form where is °C, is °F, is a fraction and is a whole number, from the conversion table above, work out what this equation is. Hint: Start with the case for 0°C to find the value of .
  2. Using this equation, what is the temperature in °F when it is 32°C?
  3. What is the temperature in °C when the temperature is 90°F?
  4. For every temperature in °C, how many corresponding temperatures are there in °F.
  5. Draw a graph to represent the relationship between °C and °F.
  6. Use your graph to work out what 35°C is in °F.

1. When a ball is thrown up into the air, it follows a path that can be described using the equation where is the height of the ball above the ground in metres and is the time since the ball was thrown, in seconds.
   1. How high is the ball after 3 seconds?
   2. When will the ball hit the ground again?
   3. What is the ball’s greatest height above the ground?
   4. Can the ball be at more than one height at any particular time?
   5. Are there any times when the ball is at the same height?

#### Guided Reflection

1. In this part of the activity, you measured the diameter and circumference of various differently sized circles.
   1. You should have noticed that all the answers in the bottom row of your table were about 3.14; in other words, in each case, the circumference was about 3.14 times the diameter. Here is a table with the values someone measured

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Lid 1** | **Lid 2** | **Lid 3** | **Lid 4** |
| Diameter | 2cm | 2.4cm | 5.2cm | 10.7cm |
| Circumference | 6.3cm | 7.5cm | 16.4cm | 33.9cm |
|  | 3.15 | 3.13 | 3.15 | 3.14 |

All the answers are more or less the same and are all about 3.14.

* 1. This number is more or less pi ().
  2. Because the ratio of the length of the circumference and the diameter of a circle is a constant number, we could predict the circumference of a lid if we know its diameter.
  3. We know that . Therefore, . Therefore, if the diameter is 12cm, the circumference will be .
  4. We know that . Therefore, . Therefore, if the circumference is 15.24cm, the diameter will be.
  5. No. There is only one possible corresponding circumference for each diameter.

1. In this part of the activity, we worked in right-angled triangles. Now the angles in the triangles you drew were probably different to the angles in the triangles we drew but this does not matter, as you will see.
   1. Here is the table we completed for the triangles we drew. Our triangle had a 60° angel and we used this one as our reference angel.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Triangle 1** | **Triangle 2** | **Triangle 3** | **Triangle 4** |
| Length of side next to the angle (adjacent) | 4cm | 6cm | 10cm | 14cm |
| Length of hypotenuse | 8cm | 12cm | 20cm | 28cm |
|  | 0.5 | 0.5 | 0.5 | 0.5 |

No matter what angles were in your triangles, you should have noticed that all the answers in the bottom row of your table were the same.

* 1. We can express the relationship between the length of the side adjacent to our reference angle and the length of the hypotenuse as . In our case, that constant was 0.5.
  2. If the length of the adjacent side is 45km, we can work out the length of the hypotenuse by rearranging our equation.

In our case, this constant was 0.5. In your case it was probably something different.

* 1. In the same way, we can use the equation for the relationship between the length of the adjacent side and the hypotenuse to work out the length of the adjacent side if we know the length of the hypotenuse.

In our case, this constant was 0.5. In your case it was probably something different.

* 1. Based on the equation that relates adjacent and hypotenuse, for every length of the adjacent side, there is only one length of hypotenuse.

In this example and the previous one, we discovered a relationship between two different variables – between diameter and circumference and between the lengths of the side adjacent to an angle and the hypotenuse in any right-angled triangle with the same angles. In the first example, this ratio is always the same and is always equal to pi. In the second example, this ratio depended on the size of the angles in the triangles.

These are both examples of functions – an equation that relates one variable to another and that allows us to predict what one of the variables will be if we know the other. But this is not all there is to functions as we will see.

1. In this part of the activity, we used an online calculator to help us convert between hectares and acres both of which are measures of area.
   1. Here is the completed table. Your table should have the same values in it.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Hectares** | 1 | 2.02 | 5 | 4.86 |
| **Acres** | 2.47 | 5 | 12.36 | 12 |

* 1. To work out this equation, we could see if there was a constant ratio between hectares and acres. It turns out that is always equal to 2.47. Therefore, we can say that

.

* 1. 150 acres in hectares: .
  2. No. you can only ever get one answer when converting from one unit to the other.
  3. Here is an image of all the points plotted.

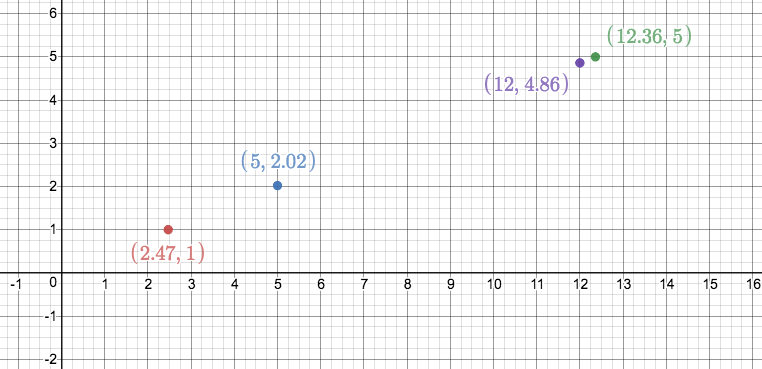
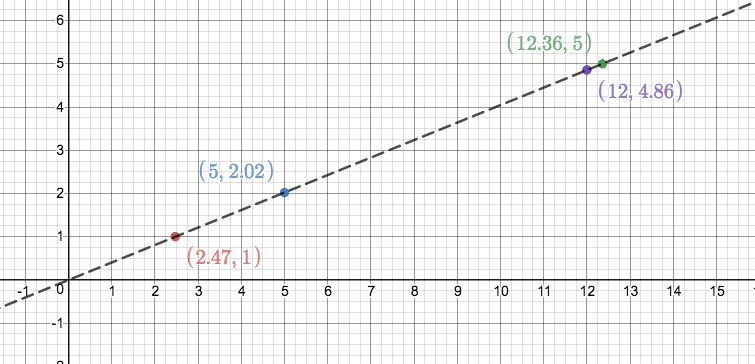
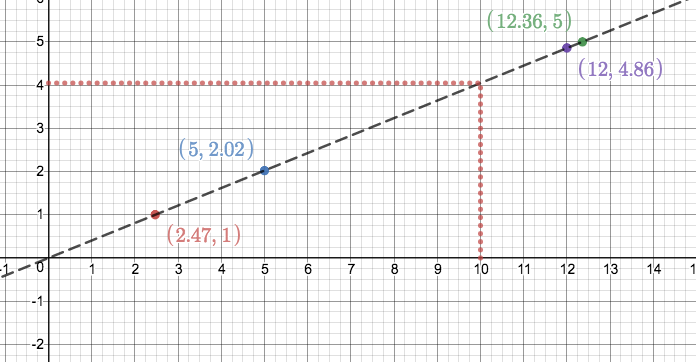


Image source: <https://www.desmos.com/calculator/ewotofu5ow>

All the points line on a straight line as shown in the image below.



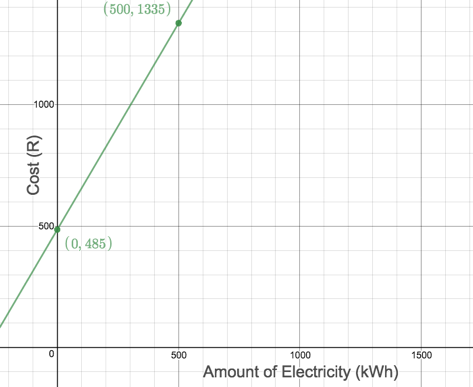
To work out how many hectares 10 acres is, you need to find 10 on the x-axis, go up to the graph and then move sideways to find the corresponding value as shown in the image below. Ten acres is about 4 hectares.



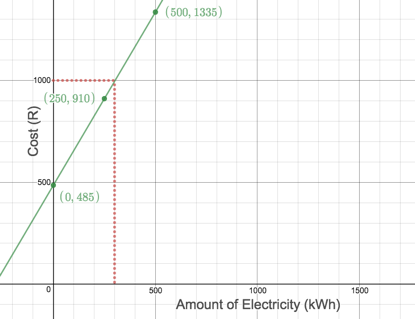
1. This question was all about the relationship between the amount of electricity a household uses and how much they then need to pay. In South Africa, electricity costs 170c per kWh (kilowatt hour) with a basic delivery charge of R485 per month.
   1. We know that for every kWh, they will have to pay 170c. Therefore, they will have to pay . But there is also a basic delivery charge of R485 per month. So their total bill will be for the month.
   2. We know that for every kWh, they will have to pay 170c. Therefore, they will have to pay . But there is also a basic delivery charge of R485 per month. So their total bill will be for the month.
   3. If they don’t use any electricity, they will still have to pay R485 per month.
   4. If we let the amount of electricity used be and the total cost (in Rands) for a month be then the equation will be .
   5. If they can only spend R600 on electricity, then and we need to solve for .

But .

* 1. Here is the graph of the relationship between cost and kWh. To draw the graph, we used three points – when , and . All these points fell on a straight line so we joined them with a solid line to represent all the possible relationships between amount of electricity and cost.



* 1. To work out how many kWh a household would use if their bill was R1,000, we start on the y-axis at R1,000 and move across to the graph and then down to see how much electricity this is. This is about 300kWh.



* 1. Here, we had a slightly more complicated relationship between our variables. We were able to express total cost as a function of the amount of electricity used. There are other functions that are similar like the cost of using a mobile phone. The more data you use, the more you pay. Or the amount of petrol you buy. The more you buy the more you pay. Or the distance you can travel on a tank of petrol. The more petrol you have, the further you can travel.

1. Functions are used every day to do simple conversions between different units.
   1. From the table of values, we can see that 0°C = 32°F. So, if our equation is of the form it means that

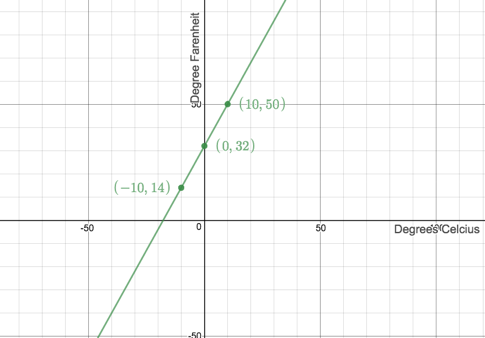
So, we know that our equation is . We can insert any pair of °C and °F values into the equation to find . Let’s use 10°C and 50°F.

Therefore, the equation is .

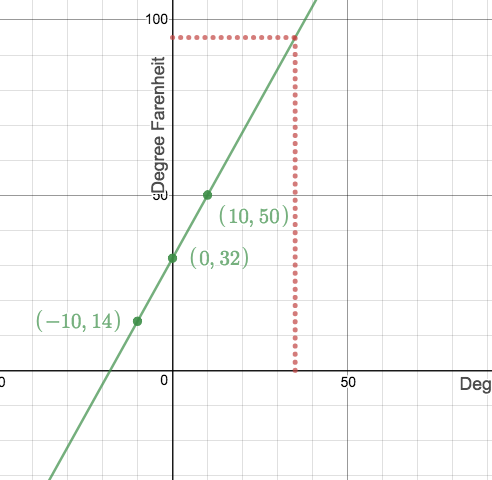
* 1. When it is 32°C, it is .

When , then .

* 1. As with all the examples we have looked at so far, there is only one answer for each °C temperate we plug into the equation.
  2. Here is the graph that represents the relationship between °C and °F.



* 1. 35°C is 95.



1. Sometimes that relationships between different variables can be quite complicated.
   1. To find out the height of the ball after 3 seconds, we have to substitute into the equation.

So when

After three seconds, the ball is 56m above the ground.

* 1. To work this out, we need to realise that when the ball hits the ground, its height is zero i.e. .

(divide through by )

(because the quadratic does not factorise nicely, we use the quadratic formula – we could have also used the completing the square method)

or

or

The negative answer does not make sense in our context, so we can ignore it. The ball will hit the ground again 4.12 seconds after being thrown.

* 1. One way to try and answer this question is through trial and error. If we know that the ball hits the ground after about 4 seconds, that means it is probably at its greatest height at about 2 seconds. So, we can create a table of values to see what this height is.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| (s) | 1.8 | 1.9 | 2 | 2.1 | 2.2 |
| (m) | 71.36 | 71.84 | 72 | 71.84 | 71.36 |

It looks like the ball reaches it maximum height at 2 seconds and this maximum height is 72m.

* 1. At any particular time, the ball can only be at one height. This makes sense if you think about it. The ball can never be, say, 25m AND 30m above the ground at the same time.
  2. We can see from the table above that the ball can reach the same height at different times. Again, this makes sense if you think about it. If you throw a ball up in the air, there will be two times when the ball is at the same height -once on the way up and once on the way down.

In each of the examples in Activity 1, we saw that we had a relationship between two variables. Here is a summary.

|  |  |  |
| --- | --- | --- |
| **Variable 1** | **Variable 2** | **Function** |
| Circumference (c) | Diameter (d) |  |
| Adjacent (a) | Hypotenuse (h) | , where |
| Degrees Fahrenheit (F) | Degrees Celsius (C) |  |
| Cost (C) | Usage (U) |  |
| Height (h) | Time (t) |  |

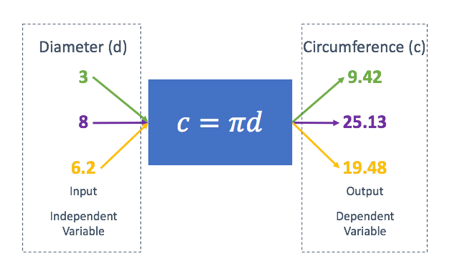
In each example, we can see that there is a **dependent** variable and an **independent** variable. For example, in the example of the height of the ball, we can see that the height of the ball **depends** on the amount of time that has passed. The height is the dependent variable and the time is the independent variable (the time does not depend on the height – it is independent of height).

In the electricity example, the cost **depends** on how much you use. The cost is the dependent variable and the usage is the independent variable.

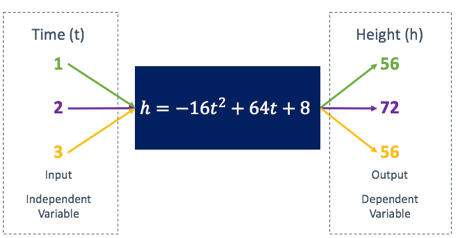
In these examples we say that **height is a function of time** and that **cost is a function of usage**.

From our examples, we can say that **circumference is a function of diameter** (the circumference depends on the diameter) and so circumference is the dependent variable and diameter is the independent variable. Or, we can say that °F is a function of °C - °F is the dependent variable and °C is the independent variable.

We can think of functions as special number machines that take inputs and give back outputs. The function number machine for circumference and diameter, takes the input of diameter, multiplies this by and returns the circumference.



The function number machine for the ball’s height and time, takes the input of time, does lots of calculations with it and gives back the height of the ball.



Each machine is different, but they do the same thing – they take independent variables (or inputs) and give back dependent variables (or outputs). We call the diagrams above **mapping diagrams**.

Activity to help learners understand that the graph of a function is a picture of all the points that satisfy the equation of the function. When we are dealing with all real numbers, there are an infinite number of these points and they are infinitesimally close to each other that the appearance is a solid line.

### Activity 2: Functions and Relations

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to understand the difference between functions and relations. |
| Stopwatch | Suggested Time You will need about 15 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

1. The relationship between and is given by . Use this equation to complete the following table.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Inputs ( | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| Outputs () |  |  |  |  |  |  |  |

1. How many output values () does each input value () generate?
2. Draw the mapping diagram for this relation using the same inputs as in the table.
3. The relationship between and is given by . Use this equation to complete the following table.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Inputs ( | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| Outputs () |  |  |  |  |  |  |  |

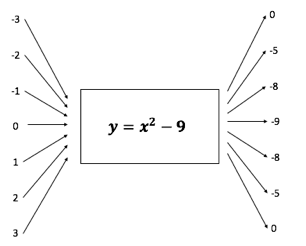
1. How many output values () does each input value () generate?
2. Draw the mapping diagram for this relation using the same inputs as in the table.
3. What is the difference between these two relationships between and ?

#### Guided Reflection

1. Here is the completed table.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Inputs ( | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| Outputs () | 0 | -5 | -8 | -9 | -8 | -5 | 0 |

1. Each input value () generates only a single output value ().
2. Here is the mapping diagram.



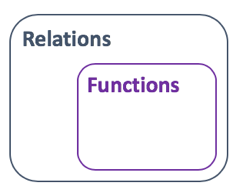
1. Here is the completed table.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Inputs ( | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| Outputs () | 0 |  |  |  |  |  | 0 |

1. Most of the input values generated two output values.
2. It is not possible to draw a mapping diagram because some inputs produce **more than one** output.
3. The first relationship only ever gave a single answer or output value for each input value. The second relationship often gave two different answers or output values for each input value.

In Activity 2 we saw that the relationship between and given by only ever produced **a single output** from every input. However, the sometimes produced two outputs from certain inputs. Getting two outputs for a single input can be confusing. Which answer do we choose? We prefer to **always** only get a single definite output for each input.

It is for this reason that we say that is NOT a function. It is a **relation** because it still describes the relationship between and . Only those relations that only ever give a single output for each input are called functions. This means that every function is a relation but not every relation is a function.



We can now fully define functions.

**A function is a mathematical relationship between two variables (the independent and the dependent variable), where every input variable only has a single output variable.**

Look back at the examples form Activity 1. Are all of these relations functions?

You will see that they are all functions. In no case does an input ever result in more than one output. It does not matter that more than one input can produce the same output. It only matters that there is only one answer every time we put an input into the machine.

Now that we know what a function is, let’s see the different ways we can represent them.

We have a very simple test to check if a relation is a function or not. It is called the vertical line test and we apply it to the picture or graph of the relation. Move a ruler or any other vertical line across the graph form left to right. It, at any point, the vertical line cuts the graph more than once, it means that for at least one input ( value), the relation gives more than one output ( value). Here are some examples.

[Examples of vertical line test]

### Activity 3: Ways to Represent Functions

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to understand the different ways we have of representing functions. |
| Stopwatch | Suggested Time You will need about 15 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

Have a look at this relation: .

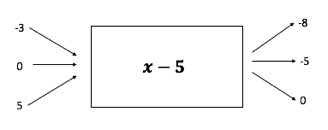
1. It this relation a function?
2. Draw a mapping diagram using the variables .
3. Complete the table with these same input variables.

|  |  |  |  |
| --- | --- | --- | --- |
| Input variable ( |  |  |  |
| Output variable () |  |  |  |

1. Write each input and output pair as ordered pairs.
2. Plot these ordered pairs on a Cartesian plan and join the points with a line.
3. What kind of graph is this?
4. What kind of equation is ? Why do you think it is called this?

#### Guided Reflection

1. Each input will only ever produce a single output. Therefore, the relation is a function.
2. Here is the completed mapping diagram.



1. Here is the completed table.

|  |  |  |  |
| --- | --- | --- | --- |
| Input variable ( | -3 | 0 | 5 |
| Output variable () | -8 | -5 | 0 |

1. Here are the three ordered pairs.
2. Here is a Cartesian plan with these three points plotted and joined with a line.

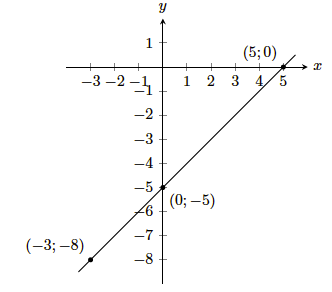


Image source: Everything Maths Grade 10 page 148

1. This is a straight line.
2. is a linear equation. It is called a linear equation because it produces a straight-line graph. Linear is another word for straight line.

All the representations of in Activity 3 are possible ways to represent a function. Of these, the most important and useful is probably the graph (the graphical representation).

However, there is one more way we have of representing functions. It is called **function notation**. We can represent as . We read this “f of x is equal to x minus 5”.

Function notation is really useful because it allows us to name different functions different things. For example, we could have

Now, when we say “the function g” or “the function h of x” we know exactly which function we are referring to. This is much better than having all these functions written as and us not being able to tell them apart. We are like parents giving their children names!

Function notation is also great for representing different function values. Take the function above. We can write (f of three). All this means is “what is the value of the function when ?”

In the same way

Also .

Watch the video [What is a Function](https://www.youtube.com/watch?v=GbQl0Ve_cH4) (04:30) for a great summary of everything we have learnt.

(<https://www.youtube.com/watch?v=GbQl0Ve_cH4>)

## Unit 2: Linear Functions

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Define what a linear function is;
2. Sketch linear functions of the form ;
3. Explain the effects on the shape of the graphs of linear functions of and ;
4. Describe the various characteristics (e.g. domain and range) of graphs of linear functions;
5. Determine the equation of linear functions from their graphs; and
6. Interpret the graphs of linear functions to make arguments or predictions.

#### Introduction

We already know what linear equations are. Remember, these are equations where the highest power on the unknow is one. Linear functions are simply the relationships between input and output variables described by a linear equation. They have the general form of or where and are constants.

In the previous unit, we saw that linear functions result in straight-line graphs.

The best way to learn about linear functions is to play with them.

### Activity 1: Sketching Linear Functions Part 1

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to understand the different ways we have of representing functions. |
| Stopwatch | Suggested Time You will need about 25 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

Have a look at the following linear functions:

1. Complete the following table of values for each function.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

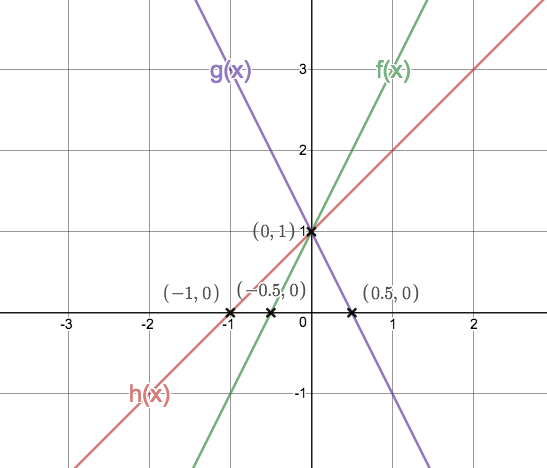
1. What is the least number of points you need in order to plot a linear function?
2. Which do you think are the easiest and most convenient points from the table of values with which to plot each function?
3. On the same set of axes, plot all three functions. You can choose any points from the table to plot each function.
4. What is the same about each function and what is the same about each graph? Remember that the general form of the linear function is .
5. What is different about each function and what is different about each graph?
6. Now visit <https://www.desmos.com/calculator/anmz0bgkes>. Here you will find a linear function in its general form with sliders that let you change the values of and .
   1. Change the value of . What effect does this have on the graph?
   2. What can you say about the value of in ?
   3. Change the value of . What effect does this have on the graph?
   4. What is the difference between a large value of like 5 and a small value of like ?
   5. What is the difference between positive and negative values of ?
   6. What does a value of mean? Why is this?

#### Guided Reflection

1. Here is the completed table.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

1. To plot a straight line, you need at least two different points.
2. The easiest points to use are those where either the or value of the point is zero. These points lie on an axis and are where the graph cuts or **intercepts** the axis. For example, for , the point is the point where the graph intercepts the x-axis.
3. Here are all three functions plotted on the same set of axes. The black crosses indicate the points where the graphs cut or intercept each axis.



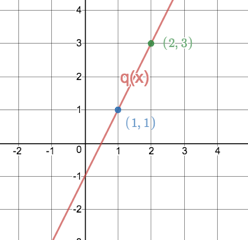
<https://www.desmos.com/calculator/hrxfkhkq7l>

1. Each function has the same value of which is . Each graph intercepts the y-axis at this same value.
2. Each function has a different value of . Each graph has a different slope or gradient.
3. Changing the variables in a function and seeing what affect they have on the shape and position of the graph of the function is a great way of getting to know a function better.
   1. As we change the value of , the graph moves up and down. When we *increase* the value of , we move the graph *up*. When we *decrease* the value of , we move the graph *down*.
   2. The value of in is the point where the graph cuts or intercepts the y-axis. We call this point the **y-intercept**.
   3. As we change the value of , we change the slope or gradient of the graph.
   4. A large value of gives a graph with a very steep slope. A small value of gives a graph with a very shallow slope.
   5. When is *positive*, the graph slopes *up* from left to right. When is *negative* the graph slopes *down* from left to right.
   6. When , the graph is a flat horizontal line. When , the function becomes . In other words, it makes no difference what input value we choose, the output value will always be . Thus, all the points on the graph will have a coordinate of .

We learnt quite a lot from that last activity. Let’s recap.

1. Sketching a linear function:

One way we have of sketching a linear function is to find and plot any two points. We can do this by choosing any two values of and then finding out what the corresponding function values are. For example, if we had , we might choose and . Then and . We would then plot the points and and join them with a straight line.



<https://www.desmos.com/calculator/w3g3bbkta2>

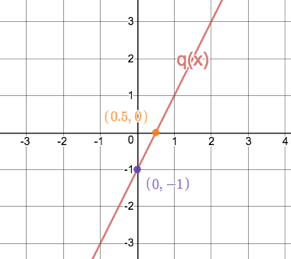
1. The dual-Intercept method:

However, we also saw that we can make our lives a bit easier if we chose to plot the points where the graph intercepts with the axes. Where the graph crosses the x-axis, this is the **x-intercept** and the y-coordinate of the point is zero and where the graph crosses the y-axis, this is the **y-intercept** and the x-coordinate of the point is zero.

So, plotting :

x-intercept (choose ): . Therefore, the point is .

y-intercept (choose ): . Therefore, the point is .



<https://www.desmos.com/calculator/gxjbcceor3>

Can you see why see can just look at the value of as a short-cut for finding the y-intercept? Can you also see why it is called the dual-intercept method for sketching a linear function? We find both intercepts.

1. The values of and

We have already seen that when the linear function is in the form the value of immediately tells us what the y-intercept of the graph is.

The value of tells us what the slope or gradient of the graph is. The bigger the value of , the steeper the graph.

Here is a summary of what we know so far:

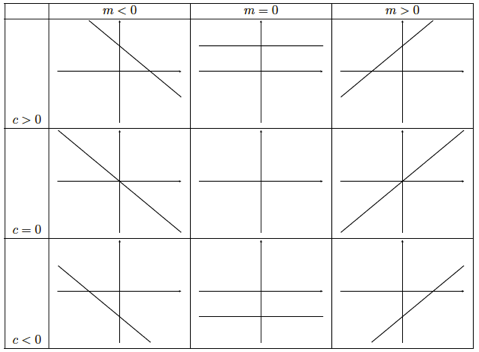


Image source: Everything Maths Grade 10 page 152

Remember, that we sometimes write the standard for of the linear function as . In this case tells us what the y-intercept is and the value of gives us the gradient of the graph.

Let’s take a closer look at the gradient.

### Activity 2: The Gradient of the Straight Line Graph

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to better understand what the gradient of a straight line is and how to measure it. |
| Stopwatch | Suggested Time You will need about 30 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

Open the interactive simulation at <https://www.desmos.com/calculator/nmfwlna6jp>. Here you will find a linear function in the form with sliders so that you can change the values of and . You will also see two points on the graph that you can drag with dotted lines that meet to form a right-angled triangle.

1. Set the graph to represent the linear function . Now move the red point to and the blue point to . You may need to click and graph on the Cartesian Plane to move these points into view.
   1. Measure the length of the horizontal purple line. How does this length relate to the values of the coordinates of the two points?
   2. Measure the length of the vertical green line. How does this length relate to the values of the coordinates of the two points?
   3. Calculate the value of . What characteristic of the graph does this value measure?
   4. Now, move the red and blue points to any other parts of the graph and recalculate the value of . What do you notice? What do you notice about this value and the value of ? What do we already know about .
   5. Change the value of . Does this change the value of or ?
2. Now set the graph to represent the function . Move the red point to and the blue point to .
   * 1. What do you expect the value of to be? Measure the length of the lines and calculate the value. Is it what you expected?
3. Set the graph to represent and move the red point to and the blue point to .
4. What do you expect the value of to be? Measure the length of the lines and calculate the value. Is it what you expected?
5. What changes to your calculations did you have to make to arrive at the answer of for ?
6. Move the green and purple points to any other parts of the graph to check your calculations for the gradient.
7. Change the value of to any other value you like and verify that you are able to use the same method as above to measure or calculate the gradient.
8. Write a general expression that will allow you to measure the gradient of any straight-line graph.

#### Guided Reflection

1. We were first asked to set up the linear function so that it was .
2. The length of the horizontal purple line is 2 units. This is really just the difference in the coordinates of the two points (.
3. The length of the vertical green line is 4 units. This is just the difference in the coordinates of the two points (.
4. . This is a measure of how steep the graph is; in other words, the gradient.
5. No matter where we move the red and blue points to on the graph, the value of . For example, if we moved the red point to and the blue point to , the value of .

The value of is the same as the value of . We know that describes the gradient of the graph. Therefore is a measure of the gradient of the graph as well.

1. Changing the value of , only moves the graph up and down. It does not change the value of or, therefore, the gradient of the graph.
2. The next linear function we investigated was .
3. Based on part 1), we expect the value of . Therefore, we expect it to be If we measure this value we get , just what we expected.
4. The next function we investigated was .
5. We expect the value of . If we do the calculations, we get , just what we expected.
6. No changes were necessary. We still just subtracted the coordinates of the two points and divided this by the difference between the coordinates of the two points.
7. If we use the points and , we get .
8. Let’s use the function and the points and .
9. If we have any two points on a straight line, say and , then we can calculate the gradient of the straight line as follows:

It does not matter which is point 1 and which is point 2, so long as you **subtract the coordinates in the same order**.

The expression for the gradient of a straight line above is important and you will need to use it often in many different scenarios. Other ways you may see the gradient of a straight line expressed are

or or

All these mean the same thing. **When moving from left to right between any two points**, how much does the graph goes up (or down) from the one point to the next, divided by how far the graph has to move to the right to achieve this up or down change.

Now that we know what gradient is and how to measure it, we can look at another way to sketch linear functions.

### Activity 4: Sketching Linear Functions Part 2

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to sketch linear functions using the gradient-intercept method. |
| Stopwatch | Suggested Time You will need about 20 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

Tasks

1. Sketch the function using the gradient-intercept method i.e. starting at the y-intercept, measuring out the gradient to find another point on the line and then join these points with a straight line.
2. Sketch the linear function that has a y-intercept at and a gradient of .
3. A straight line is parallel to the line and passes through the point . What is the equation of the line? Sketch the line.

Guided Reflection

1. We were asked to sketch the function using the gradient-intercept method. First, we need to get our function into standard form.

Now we can see that the y-intercept is at 1 and the gradient is (or ). Starting at the y-intercept, we measure out the gradient – one unit right and then two units down.

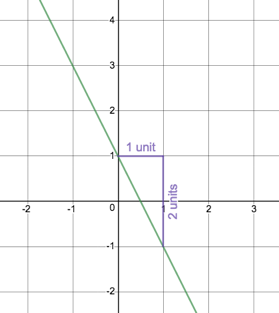


Image source: <https://www.desmos.com/calculator/b8spbmgrln>

1. We know that the gradient of the line is . In other words, if we move two units to the right, we have to move three up. Th y-intercept is the point so this is a good place to start measuring the gradient from. Here is what your graph should look like.

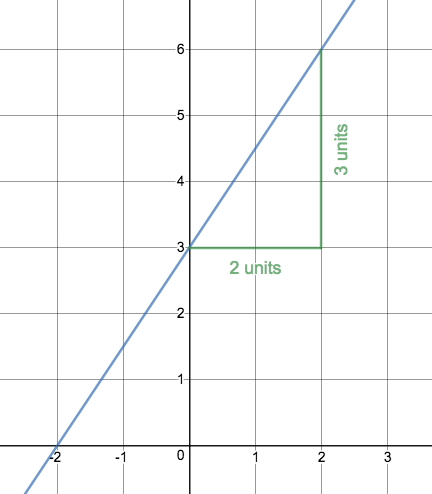


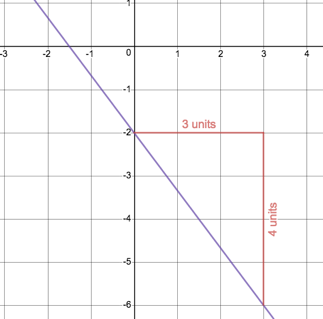
Image source: <https://www.desmos.com/calculator/5yz9hzidai>

1. We are told that our graph is parallel . Therefore, it has the same gradient. We are also told that our graph passes through the point .

To find the gradient, we have to get the parallel line into standard form.

The gradient of our graph is . Starting at the point (which is the y-intercept but could really be any point on the line), we measure out the gradient – three units to the right and then then 4 units down.

The graph looks like this.



### Activity 5: Domain and Range

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you understand what domain and range are and how to read them from a straight-line graph. |
| Stopwatch | Suggested Time You will need about 25 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

Tasks

Open the interactive simulation at <https://www.desmos.com/calculator/gt7jvcqd3w>. Here you will find the linear function with sliders so that you can change the values of and . By now, you know exactly what effect changing these sliders will have on the graph.

But there are two extra sliders as well - for and . In this activity we are going to find out what these do.

1. At the moment, the graph looks like a normal straight line. Now zoom out. You can either use your mouse or fingers or the + and – buttons on the right. At some point, you will see that the graph stops. At what value does it stop on the left? At what value does it stop on the right? Why do you think this is?
2. Now move the slider to be -20. What happens to the graph? Move the slider to be 15. What happens to the graph?
3. Now change the value of to 2. What happens to the graph? Change the value of to . What happens to the graph?
4. Change the sliders so that the graph exists between and . Work out between what corresponding values the graph exists.
5. You may have noticed that after the linear function we have the expression . This tells us that the graph may only exist where is between the values of and . Write this expression for the current values of and . Write a similar expression for the values. What do these expressions mean?
6. Now click the empty circle in row 6 on the left-hand side of the screen. The shaded area shows the or input values the graph is allowed to take. We call this the **Domain**. Now click on the empty circles in row 7 and 8. The shaded area shows the corresponding or output values the function can generate. We call this the **Range**. For the function , Set up the Domain so that the Range is [. Write the Domain as .

Guided Reflection

1. The graph stops at on the left and at on the right. These are the same values as and . Therefore, and are limiting the extent of the graph somehow.
2. If we change the value of to -20, the graph changes to stop at -20. If we change the value of to 15, the graph changes to stop at 15.
3. Changing the values of and change the gradient and y-intercept of the graph as expected but the graph still stops at -20 and 15.
4. To make the graph exist between -10 and 35, we have to make and .
5. . This means that has to be greater than or equal to -10 and at the same time less than or equal to 35. In other words, has to be between -10 and 35.

. This means that is always greater than or equal to and at the same time less than or equal to 7. In other words, can only be between and 7.

1. The Domain needed to create this Range is -.

We saw in this last activity that we can restrict the input values a function is allowed to have and, in so doing, to restrict the output values that a function can produce. We call the set of allowed input values the **Domain** and the corresponding set of possible output values the **Range**.

Sometimes the nature of the function itself, automatically restricts what input or values are allowed. Look at this example:

We know that we are never allowed to divide by zero which means the denominator in this function may never be zero. This means that . In this case, the function is allowed to take any number as an value **except** 1. We can write this Domain formally like this:

Domain:

This is called **set notation** and this expression is says that the domain is the set of values where can be any real number but may not by 1.

The Domain from part 6) of Activity 4 above, would be written formally like this:

Domain:

This says that the domain is the set of values where can be any real number between and including and . We could also write this slightly less formally using **interval notation** as

Domain: [

The square brackets tell us that the domain **includes** the end values. This method is slightly less formal and complete, because it does not explicitly tell us that can be any real number within this range.

Generally, though, unless there is a specific restriction imposed, linear functions can take any real number as an input and it is possible to generate any real number as an output. Therefore, the Domain and Range of linear functions is always

* Domain: or - the round brackets, indicate that the Domain does not include or .
* Range: or - the round brackets, indicate that the Range does not include or .

### Activity 6: Interpreting Linear Functions

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you gain experience interpreting linear functions. |
| Stopwatch | Suggested Time You will need about 25 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

Tasks

Simultaneous equations

Finding equations - <https://www.youtube.com/watch?v=rzUJatVndcY>, https://www.youtube.com/watch?v=3IPXiZQt6Ug

General interpretation

### Activity 7: End of Section Questions

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you consolidate your knowledge and understanding of linear functions. |
| Stopwatch | Suggested Time You will need about 25 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

Tasks

Questions from Everything Maths Grade 10 Exercise 6-2 (pg 156)

## Unit 3: Quadratic Functions

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Define what a linear function is;
2. Sketch quadratic functions of the form , and ;
3. Explain the effects on the shape of the graphs of quadratic functions of , and ;
4. Describe the various characteristics (e.g. domain and range) of graphs of quadratic functions;
5. Determine the equation of quadratic functions from their graphs; and
6. Interpret the graphs of quadratic functions to make arguments or predictions.

#### Introduction

You may remember when we first came across quadratic trinomial expressions in Sub-Topic 1, we said they are everywhere. Remember that quadratic trinomial expressions are expressions where the highest power on the variable is 2 and that they have the general form of where , , and are constants.

We also spent quite a bit of time in Sub-Topic 2 learning how to solve quadratic equations. All of that work was largely to allow us to work with quadratic functions.

The general form of a quadratic function that you will most often see is . But there are two other, simpler forms that we will also work with in this unit.

Of all the functions we will look at, quadratic functions are probably the most important. They produce graphs that we call **parabolas**. Parabolas crop up everywhere. If you cut a satellite dish in half through the centre, you get a parabola. The same is true for nearly all mirrors used in telescopes, headlamps and some light bulbs.

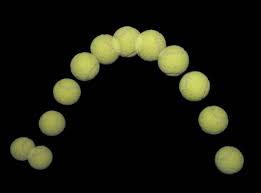


https://cdn.pixabay.com/photo/2012/12/17/15/22/satellite-70409\_960\_720.jpg

The ropes or chains hanging from the pillars of suspension bridges follow a parabolic curve. When you throw a ball (or any object) through the air, its path is a parabola.

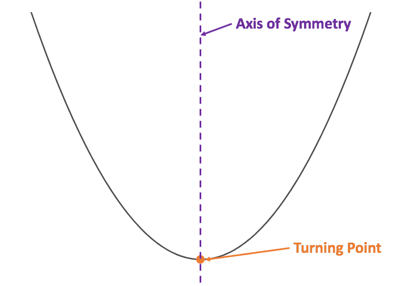


<https://upload.wikimedia.org/wikipedia/commons/2/2a/Golden_Gate_Bridge_Dec_15_2015_by_D_Ramey_Logan.jpg>



<https://encrypted-tbn0.gstatic.com/images?q=tbn:ANd9GcQLlWdFfCDTqj6qNQIg4uksD2RJWk-bKUa05D9TMfoCEWgzF8Yt>

All of these examples show us the general shape of the parabola. There are two important characteristics of parabolas. The one is that they have a **turning point** – a point where they turn or change direction. The other is that they are symmetrical about the vertical line that goes through their turning point. This means that, if you folded a parabola along this vertical line, the two arms would be exactly on top of each other. We call this line the **axis of symmetry**.



Watch the video [Parabolas Intro](https://www.khanacademy.org/math/algebra/quadratics/parabolas-intro-alg1/v/parabolas-intro) (08:15) for a good introduction to parabolas.

(<https://www.khanacademy.org/math/algebra/quadratics/parabolas-intro-alg1/v/parabolas-intro>)

Just a note – in the video the turning point is called the **vertex**. Turning point and Vertex mean the same thing when we are talking about parabolas.

### Activity 1: Sketching Quadratic Functions Part 1

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to sketch simple quadratic functions to start to understand some of their basic properties. |
| Stopwatch | Suggested Time You will need about 40 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. Have a look at these two functions:
   1. Which function is the linear function? Why?
   2. Which function is the quadratic function? Why?
   3. Plot both functions on the same set of axes. You can see any method you like to plot the linear function. Use the following table of values to help you plot the quadratic function. Plot the points and then join them with as smooth a curve as possible.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|  |  |  |  |  |  |  |  |

* 1. What is the same about each function? What is the same about each graph?
  2. What is different about each function? What is different about each graph? Think particularly about the intercepts and what you know about the number of roots we get for linear and quadratic equations.
  3. If vertical lines have the general form of where is a constant, what line is the axis of symmetry of the quadratic function?
  4. What is the turning point of the quadratic function?
  5. What is the Domain and Range of the linear function?
  6. What is the Domain and Range of the quadratic function?

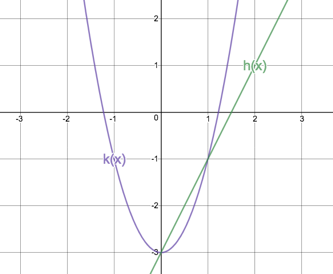
1. Visit the interactive simulation at <https://www.desmos.com/calculator/ylpnsabdg4>. Here is a quadratic function in the form of .
2. How is this form different from the general form of the quadratic function ?
3. Change the value of . What effect does this have on the graph? How does this relate to how affects the function ?
4. Change the value of . What effect does this have on the graph? How does this relate to how affects the function ?
5. What is the difference between a large value of like 5 and a small value of like ?
6. What is the difference between positive and negative values of ?
7. What does a value of mean? Why is this?

#### Guided Reflection

1. We were given two similar looking but different functions.
2. is the linear function. The highest power on is 1. The function is in the form of a linear equation.
3. is the quadratic function. The highest power on is 2. The function is in the form of a quadratic equation.
4. Here is the completed table of values for .

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|  | 15 | 5 | -1 | -3 | -1 | 5 | 15 |

Here are the two functions sketched.



1. Each function has a constant term of and each graph has a y-intercept of . Each graph also has a coefficient on the leading term of 2.
2. is a linear function with a highest power of 1. It has one intercept with the x-axis (when corresponding to the fact that linear equation only have one root or answer. However, is a quadratic function with a highest power of 2. It has two intercepts with the x-axis (when corresponding to the fact that quadratic equation has two roots or answers.
3. The parabola is symmetrical about the y-axis. All the values on the y-axis are zero. Therefore, the axis of symmetry is the line .
4. The turning point (or TP) of the quadratic function is the point .
5. The domain and range of the linear function are both unrestricted. They are both all real numbers.

Domain:

Range:

1. The domain of the quadratic function is all real numbers. There are no input values (values of ) that we cannot use.

The range, however, is different. We can see from the graph that the values are never less than is the minimum value of the graph. At this point the graph turns.

So, we can write the domain and range as follows:

Domain:

Range:

1. Here we looked at a simpler form of the quadratic function.
2. The form is the same as except that the coefficient, , of the term is zero, so this whole term falls away.
3. If we change the value of , we move the graph up and down. The value of is also where the graph cuts the y-axis. In other words, tells us the y-intercept.

This is exactly the same as the effect of in the linear equation (), where is also the y-intercept.

1. Changing the value of , changes the steepness of the graph. This is very similar to how affects the gradient of the straight-line graph (). Although, it must be said that, because the parabola is not a straight line, “steepness” is a little bit different.
2. The larger the value of in , the steeper or narrower the graph becomes. The smaller the value of , the less steep or wider the graph becomes.
3. When is a *positive* number, the graph bends *upwards*. It is like a “smiley face”. When is *negative*, the graph bends *downwards* into a “sad face”.
4. When , the graph is a flat horizontal line. This is because when , becomes just , a constant. No matter what the input value is, the output will always be this same constant.

In this last activity we saw that the value of in and the simple form of the quadratic function do the same thing – they tell us where the graph cuts the y-axis. This makes sense though. Remember that one of the ways to sketch a linear function is the dual-intercept method where we find where the graph intercepts both axes.

* x-intercept (let
* y-intercept (let

Well the same principle applies to the quadratic function. We can find the y-intercept by letting and we can find the x-intercept by letting . Have a look at this example.

Suppose we have . Let’s find the intercepts with the axes.

* y-intercept (let : . As we expected, the y-intercept is simply the value of .
* x-intercept (ley : . As we expected, we have two points where the graph intercepts the x-axis. Can you see why we spent all that time learning how to solve quadratic equations?

We also learnt from the last activity that the value of in the simplified form of the quadratic function , determines whether the graph bends up into a smiley face or down into a sad face and how steep or shallow the curve is. Here is a summary.

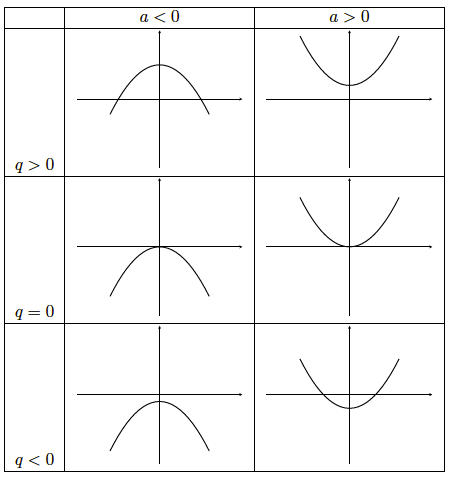


Image source: Everything Maths Grade 10 (pg 161)

Here’s a short activity to make sure you understand how the values of and affect the graph of .

### Activity 2: The Effects of and in

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help to make sure you understand how the values of and affect the graph of . |
| Stopwatch | Suggested Time You will need about 10 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

Visit the interactive simulation at <https://www.desmos.com/calculator/z1vvti3zv8>. Here you will find and several other parabolas.

1. Change the values of and for , so that it matches each of the other parabolas and write down the what each of these functions are.
2. Write down the domain and range of each function.
3. Write down the axes of symmetry (AS) and the turning point (TP) of each function?
4. What relationship exists between the y-intercept and the TP of each function? Why is this the case?
5. What relationship exists between the TP and the range of each function? How does this change when and when ?

#### Guided Reflection

1. : We know that the value of in this function must be because this is where the graph cuts the y-axis. So, we change to be and see that it fits perfectly. So

: We know that the value of must be . So, we change to be . But is steeper so the value of must be greater than 1. So, let’s make . It fits. Therefore, .

: We know that the value of must be . So, we change to be . But is steeper and bends down. Therefore must be negative and have an absolute value (its value ignoring its sign) greater than 1. So, let’s make . is still steeper so, let’s try . It fits. Therefore, .

: We know that the value of must be . So, we change to be . But bends down. Therefore must be negative. So, let’s make . It fits. Therefore, .

1. Domain: Range:

Domain: Range:

Domain: Range:

Domain: Range: – Remember that because the graph bends down, 0 is the **maximum** value the graph can reach.

Domain: Range: – Remember that because the graph bends down, 2 is the **maximum** value the graph can reach.

1. : AS: TP:

: AS: TP:

: AS: TP:

: AS: TP:

: AS: TP:

1. For each function, the y-intercept and the value of the TP are the same. This means that for each function, the y-intercept and the TP are the same point. This is because each parabola has the y-axis as its AS.
2. For each function, the TP is the limit of its range. If and the graph bends up, then the TP is a **minimum** TP and the range is the value of the TP to positive infinity. If and the graph bends down, the TP is a **maximum** and the range is negative infinity to the value of the TP.

All the quadratic functions we have looked at so far have had the y-axis as their AS and the TP and the y-intercept have always been the same point. All the functions we have looked at have been of the form . In other words, . Do you think that if the parabola might shift left or right? Let’s find out.

### Activity 3: The Turning Point Form

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to better understand what the axis of symmetry and turning point of a quadratic function are and how to find them. |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

Visit the interactive simulation at https://www.desmos.com/calculator/nrlhtce2tc. Here you will find a quadratic function with the general form of and sliders to change the values of , and .

1. Write down the current equation of the function (where , and ).
2. What is the y-intercept, AS, TP, domain and range of ? Is the TP a minimum or a maximum TP?
3. When , what is the relationship between the AS, the TP and the y-intercept?
4. What would we need to do to to make the y-intercept the point ? Make the change to see if you are right.
5. What do we need to do to make the TP of the point . Make whatever changes you need to, ensuring that the y-intercept remains the point and that the TP remains a minimum. Write down the new equation for .
6. What is the new expression for ? Does this expression still correctly predict the value of the y-intercept? Does it still correctly predict the fact that the TP is a minimum? Is there a clear and obvious relationship between the value of and the AS and TP when ?
7. What is the domain and range of this function?
8. Now complete the square of this quadratic equation to get into the form . We call this the **turning point** form. What are the values of , and ?

If you need help with completing the square watch the video [Completing the Square – Quadratics](https://www.youtube.com/watch?v=wvfZs4Rsx3Y) (03:45) to get you started. If you need more help, you should look back at Sub-Topic 2, Unit 2 Activity 2.

1. What do you notice about the coordinates of the TP and the values of and in your new expression of ? Does it make sense why we call this form the turning point form?
2. What is the new AS of ? How could we write the AS in terms of or ?
3. How can we write the range in terms of or ?
4. Using the turning point form of the quadratic function as a start, write a new expression for that would have a TP of and would have a maximum TP. Is there only one expression you can write?
5. Now expand your new expression for back into the form and alter the values of , and in the interactive simulation to see if you are correct.
6. Without changing the values of , and in the interactive simulation, work out what the TP of is. Is this a maximum or minimum TP? What is the AS?

Now change the values of, and on the simulator to see if you are correct.

#### Guided Reflection

1. .
2. y-intercept:

AS:

TP:

Domain:

Range:

The TP is a minimum.

1. When , the AS is the line . The TP is on the y-axis and, therefore, also the same as the y-intercept.
2. We have to change the expression of the function to .
3. Because the TP is now no longer on the y-axis, we should start by changing the value of . If we change the value of to 2, we get the desired TP while keeping the TP a minimum and keeping the y-intercept the point .
4. The new expression for the function is . The value of is still the y-intercept. The value of and the TP is still a minimum. There is no clear relationship between the TP and the value of now that .
5. Domain: Range:
6. Completing the square is a technique we learnt to solve quadratic equations. When we complete the square, we add whatever is necessary to the quadratic expression in order to create a perfect square when we factorise. If you would like to revise the process of completing the square, go back to Sub-Topic 2, Unit 2 Activity 2.

Our function is .

(if we add 1 then the first three terms of the quadratic will factorise as , a perfect square. But we must remember to keep the function the same by subtracting 1 as well.)

In relation to the general form , we can see that , and .

1. The coordinates of the TP are the same as the value of and . The coordinates of the TP are .
2. The new AS is the line . It is still the same as the coordinate of the TP. We can also say that the AS is the line .
3. Because the TP is a minimum, we know that the range is all values of which are greater that the coordinate of the TP. Therefore, we can write general expressions for the range as follows:

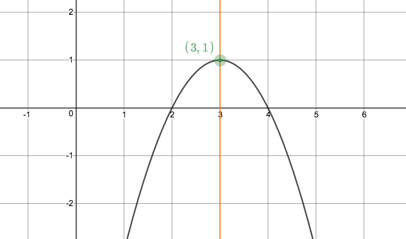
Range: or

1. The turning point form of the quadratic function is and the TP we want is . Therefore and . So, our function is of the form . But the TP is a maximum. Therefore, . Our function is thus .

Our function might also be or . In fact, could be any number less than zero. All these functions would still have a maximum TP at .

1. Let’s take the simplest version of .

This is what this function looks like.



<https://www.desmos.com/calculator/gny8z8y7pu>

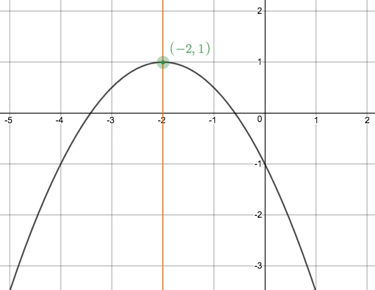
1. In order to find the TP of the given function, we need to complete the square.

(if the value of is not 1, it is always best to start by taking it out as a common factor from the terms.

(be careful here. To balance out what we added inside the bracket, we need to multiply it by the common factor outside the bracket and then change the sign. . This is why we have to balance the function with +2.

The function is now in the turning point form. . The TP is the point which is the point . so, the TP is a maximum. The AS is the line which is the line .

1. Here is the graph of this function.



<https://www.desmos.com/calculator/gny8z8y7pu>

We can clearly see that the TP is the point , that it is a maximum TP and that the AS is the line .

Wow, we learnt so much in that activity. Let’s quickly recap what we know.

1. The general form of the quadratic function is .
2. The value of is the y-intercept of the graph.
3. If , the graph has a minimum TP; if , the graph has a maximum TP.
4. When , the AS is the line and the TP and the y-intercept are the same point.
5. When , the graph is shifted left or right so that the AS is **no longer** the y-axis and the TP and y-intercept are **no longer** the same point.
6. If we complete the square, we can convert the general form () into the TP form ().
7. The TP form tells us that the TP is the point and that the AS is the line .
8. The TP form also tells us that the range can we written as
9. Range: or if (a minimum TP)
10. Range: or if (a maximum TP)
11. The TP form tells us that shifts the graph left and right (horizontal shift) and shifts the graph up and down (vertical shift).

How about a quick game to consolidate all we know?

### Activity 4: Quadratic Marbleslides

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you consolidate your knowledge of the quadratic function. |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. Visit <https://student.desmos.com/?prepopulateCode=c7ysax>. Click the **Join** button (you do not need to sign in) and complete all the screens.

#### Guided Reflection

1. Here are some answers to some of the screens.

Fix It #1: On this screen we had to change the value of from to . The function became .

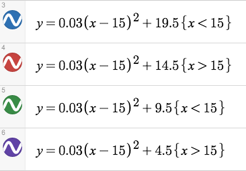
Fix It #2: Here we had to make two changes. We had to decrease the value of from 2 to about 0.5 and we had to extend the domain of the function to include values of up to 7. The new function became .

Fix It #3: Here we had to make two changes. We had to move the TP to the point and then increase the domain to include values of up to 8. The new function became .

Fix It #4: We were asked to only change one number. Changing the domain to did the trick.

Verify #4: You would need an expression similar to .

Challenge Slide #1: Here is one possible solution. What did you get?



Challenge slide #2: Here is one possible solution - .

We now know enough about quadratic functions to be able to plot any quadratic function.

### Activity 5: Sketching Quadratic Functions Part 2

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you consolidate your knowledge of the quadratic function. |
| Stopwatch | Suggested Time You will need about 75 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

1. Have a look at this function: .
2. Find the y-intercept.
3. Find the x-intercepts.
4. Find the TP and the AS. Is the TP a maximum of a minimum?
5. Make a neat sketch of , using the above points.
6. If , show, by completing the square, that the AS and the x coordinate of the TP are given by .
7. By finding the intercepts, TP and AS, make neat sketches of the following functions on the separate sets of axes.
8. State the domain and range of each of the functions in question 3).
9. and is the same shape as except that it is shifted 1 unit to the right and 3 units down. What is the equation of ? Hint: This means that the TP of is 1 unit to the right and 3 units down from the TP of .

#### Guided Reflection

1. We were given the function . We can rewrite this as .
2. We know that the y-intercept is the value of . Therefore, the y-intercept is the point .
3. To find the x-intercepts, we need to set .

or

The x-intercepts are the points ( and .

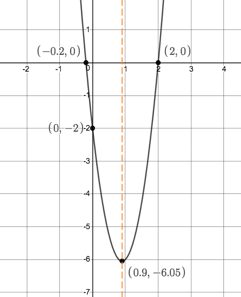
1. To find the TP, we have to get the function into the TP form.

(take out the common factor from the terms.

(add whatever is necessary to complete the square. In this case . Remember to balance this correctly.)

The TP is the point or and the AS is the line . The value of is positive, therefore, the TP is a minimum.

1. Here is a sketch of .



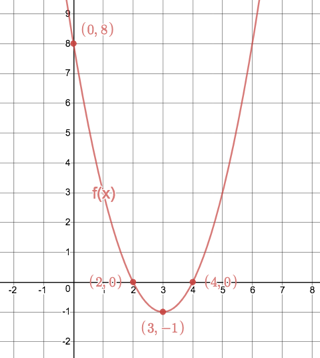
<https://www.desmos.com/calculator/35f3ghoa67>

1. Completing the square in question 1) was a bit of a mission. It would be nice if we had a more general expression for the TP that we could use. Let’s complete the square of the general for of the quadratic function.

If the TP form is , this means that and .

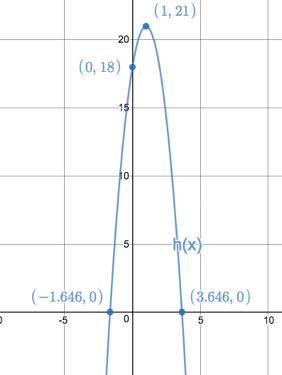
We could also find the value of by calculating i.e. substitute the value of into the function to find the corresponding output value.

1. We were asked to sketch four quadratic functions.
2. Sketch



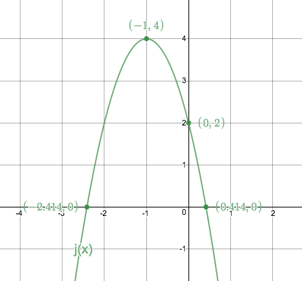
<https://www.desmos.com/calculator/ndu6is12ik>

1. Sketch



<https://www.desmos.com/calculator/ndu6is12ik>

1. Sketch



<https://www.desmos.com/calculator/ndu6is12ik>

1. The function we were given was . Because it is already in TP form, we can immediately tell what the TP and AS are. The TP is and the AS is the line . We also know that the TP is. Maximum because .

However, we might have a problem finding the x-intercepts. The TP is a maximum AND it is below the x-axis. Therefore, the graph does not cut the x-axis at all. Let’s confirm this by solving for when .

(let

(multiply through by )

We know that whenever we square any real number, the answer is always positive ( and . So, we cannot take the square root of a negative number. This confirms that there are no real solutions and, hence, no x-intercepts.

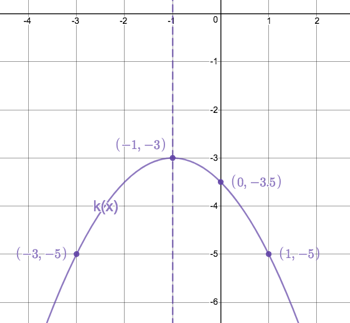
To help us get the shape of the parabola right, we need to find any two other points that the graphs goes through. Let’s use .

. So, we have the point .

Because we know the graph is symmetrical about the line , let’s use the symmetrical point 2 units to the right of the AS i.e. .

. So, we have the point . This confirms that the graph is symmetrical about the AS.

We can now sketch the function.



<https://www.desmos.com/calculator/ndu6is12ik>

1. We were asked to state the domain and range of each function in question 3). We can either read these off the sketches, or use our knowledge of the TPs.
2. Domain: or Range: or
3. Domain: or Range: or
4. Domain: or Range: or
5. Domain: or Range: or
6. There are two ways of answering this question. We can find the TP of and then work out the TP and equation of or we can make some clever substitutions into .

**By finding the TP:**

:

AS: . This is also the x-coordinate of the TP.

. Therefore, the TP of is the point .

The TP of is but the shape is the same, therefore .

**By making substitutions:**

We know that is the same shape as but has been moved 1 unit to the right and 3 units down. That means that every point on the graph of has been moved 1 unit to the right and 3 units down. Here is a table of some corresponding points on and .

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  |  |  |

We can see from the table that every x coordinate has had 1 added to it and every y coordinate has had 3 subtracted from it.

To shift the y coordinate of every point on the graph of down by 3 units is easy. We can just say that because we know that . When we input, say 1, into we would still need to subtract 3 from the answer to find the corresponding point on .

To shift the x coordinate of every point on the graph of to the right by 1 unit, we can say that . In other words, if we input, say 1, into we would need to input into to get the same answer.

But we don’t know the equation of so we have to write this relationship the other way around i.e. . For every we input into we would need to input ( into to get the same answer.

Putting these two shifts together we get that .

These kinds of transformations of graphs can be quite tricky so let’s do another activity to make sure we understand how they work.

### Activity 6: Transformations

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the various ways we can transform graphs and how these transformations can be done. |
| Stopwatch | Suggested Time You will need about 80 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. Visit the interactive simulation at <https://www.desmos.com/calculator/uzlou9sehc>. Here you will find three functions, and and along with sliders to change the values of and . Currently so and so .
2. Without doing anything, what do you think will happen to the graph of if you made ? What do you think would happen to the graph of if you made ?
3. Make the changes to see if you are right.
4. Without doing anything, what do you think will happen to the graph of if you made ? What do you think would happen to the graph of if you made ?
5. Make the changes to see if you are right.
6. If is moved 3 units down, write down the equation of the new function in terms of .
7. If is moved 3 units to the right, write down the equation of the new function in terms of .
8. If is moved 4.5 units to the left, write down the equation of the new function in terms of .
9. If is moved 3 units right and 4 unit down, write down the equation of the new function in terms of . Also, write the new function in full.
10. Now watch the video called [Transformations of Graphs: Translations](https://www.youtube.com/watch?v=7qj4kWrxTuY) (03:15) for a great summary of these **translation** transformations and some extra practice questions.

(<https://www.youtube.com/watch?v=7qj4kWrxTuY>)

1. Take the function . Make a quick sketch of the function on a piece of paper.
2. Now multiply the y coordinate of a few points on the graph by , plot these new points and sketch the resulting graph. What has this done to the original graph?
3. Write an expression for your new function in terms of . Remember, every y coordinate was multiplied by .
4. Now multiply the x coordinate of a few points on the graph by , plot these new points and sketch the resulting graph. What has this done to the original graph?
5. Write an expression for your new function in terms of . Remember, every x coordinate was multiplied by .
6. Visit the interactive simulation at <https://www.desmos.com/calculator/bssf8tgxwd> to check that your sketches of and are correct and have been expressed in terms of correctly.
7. Now watch the video called [Transformations of Graphs: Reflections](https://www.youtube.com/watch?v=r74cyHCUZQs) (01:55) for a great summary of these **reflection** transformations (reflect graphs about the x- and y-axes).

(<https://www.youtube.com/watch?v=r74cyHCUZQs>)

1. Visit the interactive simulation at <https://www.desmos.com/calculator/vtf19d1aer>. Here you will find three functions, and and along with sliders to change the values of and . Currently so and so .
2. Without doing anything, what do you think will happen to the graph of if you made ? Remember, making will multiply the y-coordinate of every point on by 3? What do you think would happen to the graph of if you made ?
3. Make the changes to see if you are right.
4. Without doing anything, what do you think will happen to the graph of if you made ? Remember, making will multiply the x-coordinate of every point on by 2. What do you think would happen to the graph of if you made ?
5. Make the changes to see if you are right.
6. is a point on . What will the corresponding point on be if ?
7. is a point on . What will the corresponding point on be if ?
8. is a point on . What will the corresponding point on be if ?
9. is a point on . What will the corresponding point on be if ?
10. Watch the video called [Graph Transformations](https://www.youtube.com/watch?v=MLIgFdxGM-o) (3:40) for a great summary of these “stretch” and “squash” transformations.

(<https://www.youtube.com/watch?v=MLIgFdxGM-o>)

1. has been transformed by . If the point was , where is it now?

#### Guided Reflection

1. The transformations that we look at in this question are called **translations** – where we move a graph around the Cartesian plane **without** changing its shape.
2. If we made we would move the whole graph 3 units *up*. This is because we would be *adding* 3 to every output value generated by .

If we made we would move the whole graph 2 units *down*. This is because we would be *subtracting* 2 to every output value generated by .

1. Here are pictures of the translated graphs.

|  |  |
| --- | --- |
|  |  |
|  |  |

1. If we made we would move the whole graph 2 units *to the right*. This is because to get the same output out of both and , we would need to *subtract* 2 from the input before we fed it into .

If we made we would move the whole graph 5 units *to the left*. This is because to get the same output out of both and , we would need to *add* 5 from the input before we fed it into . .

1. Here are pictures of the translated graphs.

|  |  |
| --- | --- |
|  |  |
|  |  |

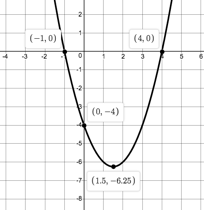
1. This is a **vertical** (up or down) shift so, in general, . is moved 3 units down, so .
2. This is a **horizontal** (left or right) shift so, in general, . is moved 3 units to the right, so .
3. This is a **horizontal** (left or right) shift so, in general, . is moved 4.5 units to the left, so .
4. This is a both a **vertical** and **horizontal** shift so, in general, . is moved 3 units right and 4 units down, so .

If , we can write the new function as

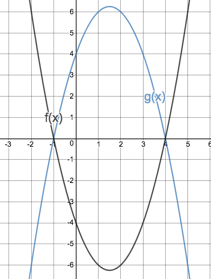
For a full worked solution of this question watch the video called [Shifting parabolas](https://www.khanacademy.org/math/algebra/quadratics/transforming-quadratic-functions/v/example-translating-parabola) (04:40).

(<https://www.khanacademy.org/math/algebra/quadratics/transforming-quadratic-functions/v/example-translating-parabola>)

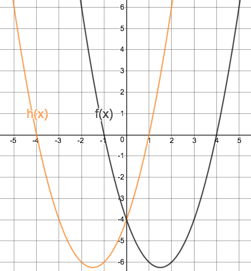
1. The transformations that we look at in this question are called reflections – where we reflect the graph about a line (in this case the x- and y-axes). Here is a picture of the graph of .



1. If we take the x- and y-intercept and the TP and multiply these y-coordinates by , we get four new points - Plotting these points and joining them gives a graph which has been flipped around horizontally, or is reflected about the x-axis.



1. If the y-coordinate of every point has been multiplied by then .
2. If we take the x- and y-intercept and the TP and multiply these x-coordinates by , we get four new points - Plotting these points and joining them gives a graph which has been flipped around vertically, or is reflected about the y-axis.



1. If the x-coordinate of every point has been multiplied by then .
2. The transformations that we look at in this question are called **scaling** transformations – where we stretch or squeeze the graph **without** moving it anywhere.
3. If we made , the y-coordinate of every point on would be multiplied by 3. This would stretch the graph out and make it “longer and thinner”.

If we made , the y-coordinate of every point on would be divided by 2. This would squash the graph and make it “shorter and fatter”.

1. Here are pictures of the transformed graphs.

|  |  |
| --- | --- |
|  |  |
|  |  |

1. If we made , the x-coordinate of every point on would have to be divided by 2 to get the same output. This would squash the graph horizontally and make it “thinner”.

Note: Horizontal transformations also seem a bit counter-intuitive. They are not what you might at first think. Remember how was a thinner graph than ? Well, if , then will be thinner than .

If we made , the x-coordinate of every point on would have to be multiplied by 3 to get the same output. This would stretch the graph out horizontally and make it “fatter”.

1. Here are pictures of the transformed graphs.

|  |  |
| --- | --- |
|  |  |
|  |  |

1. If we are dealing with a vertical scaling and every y-coordinate will be multiplied by 2. Therefore, the transformed point would be .
2. If we are dealing with a horizontal scaling and every x-coordinate will be multiplied by 2. Therefore, the transformed point would be .
3. If we are dealing with a horizontal scaling and every x-coordinate will be multiplied by 4. Therefore, the transformed point would be .
4. If we are dealing with a horizontal scaling and every x-coordinate will be divided by 3. Therefore, the transformed point would be .
5. If we are dealing with a vertical scaling and every y-coordinate will be divided by 2. Therefore, the transformed point would be .
6. We know that has been transformed by . It has undergone various transformations. Let’s work from the inside out.

We know that ) shifts the graph 4 units to the right. So moves from to .

We know that the squashes the graph vertically (all the y-coordinates are divided by 2). So moves from to .

We know that , reflects the graph about the x-axis by multiplying all the y-coordinates by . So moves from to .

Finally, we know that , translates the graph up by 3 units by adding 3 to all the y-coordinates. So moves from to .

In that last activity we learnt about three different types of transformations that we can make to graphs:

#### Translations

These move the graph vertically (up or down) or horizontally (left or right) without changing the shape of the graph.

* Vertical Shift: : If the graph moves up. If , the graph moves down.
* Horizontal Shift: : If the graph moves left. If , the graph moves right.

#### Reflections

We looked at reflecting graphs about the x- and y-axes.

* Vertical Reflection (about the x-axis or the line ): .
* Horizontal Reflection (about the y-axis or the line ): .

#### Scaling

We looked at vertical and horizonal scaling that makes the graph fatter or thinner.

* Vertical Scaling: : If the graph is stretched out vertically. If , the graph is squashed in vertically.
* Horizontal Scaling: : If the graph gets thinner. If , the graph gets fatter.

So far, we have learnt how to reflect graphs about the lines (the y-axis) and (the x-axis), but what about reflections about the line ? Let’s investigate.

### Activity 7: A Very Special Transformation

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to reflections of graphs about the line and the concept of inverse functions. |
| Stopwatch | Suggested Time You will need about 35 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

Have a look at these graphs.

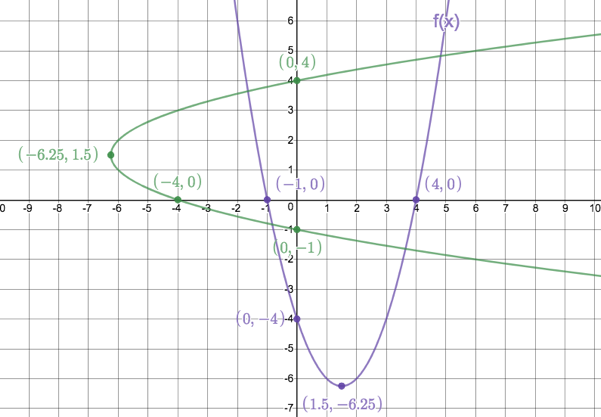


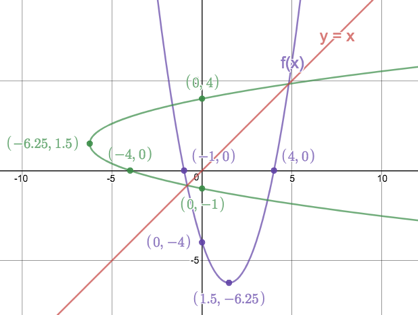
Image source: <https://www.desmos.com/calculator/4i0rocwco0>

One is the graph of .

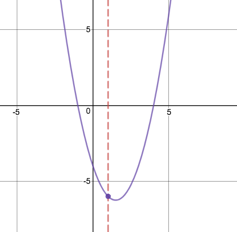
1. What do you notice about each of the labelled points on in relation to their counterparts on the other graph? Think about the TP and intercepts with the axes.
2. Draw a straight line on the graph that is the line that reflects onto the other graph. What is the equation of this line?
3. What is the AS of ?
4. What is the AS of the other graph?
5. What is the domain and range of ?
6. What is the domain and range of the other graph?
7. Is a function? How do you know?
8. Is the other graph a function? How do you know.
9. If the other graph is not a function, what could you do to make it a function?
10. If the other graph is a reflection of about the line , write down the equation of the other graph.
11. Find the value of then input this answer for into the equation of your other graph. What do you notice?

#### Guided Reflection

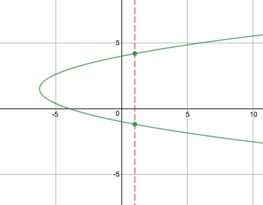
1. Each of the corresponding points on and the other graph have been swopped around i.e. what was the x-coordinate is now the y-coordinate and visa versa - . For example, the TP of is and the turning point of the other graph is .
2. The equation of the line that reflects onto the other graph is .



1. The AS of is .
2. The AS of the other graph is .
3. Domain: Range:
4. Domain: Range:
5. is a function. For every input of there is only ever one output of . This can be confirmed with the vertical line test. A vertical line will only every cut the graph once.



1. The other graph is NOT a function because there are many times when a single input generates more than one output as show by the vertical line test.



1. To make this other graph a function, we would have to get rid of one of the “arms” of the parabola. We could do this by restricting the domain of the original function to only those values of or . The excluded part of the graphs are shown with dotted lines below.

|  |  |  |
| --- | --- | --- |
| Domain of |  |  |
| Graph of and other graph |  |  |

1. If and we have swopped all the ’s and ’s around, then the equation of the other graph is .

Let in :

or

We get back our original input value as one of the answers.

From this last activity we saw that if we reflect a graph about the line , we create a graph whose equation undoes what the original function did. We call such equations **inverses**. If , the inverse of has the equation .

Watch the video [Intro to inverse functions](https://www.khanacademy.org/math/algebra2/manipulating-functions/introduction-to-inverses-of-functions/v/introduction-to-function-inverses) (09:05) to learn more about inverses.

(<https://www.khanacademy.org/math/algebra2/manipulating-functions/introduction-to-inverses-of-functions/v/introduction-to-function-inverses>)

Here is a quick summary of inverses:

1. The inverse of a function undoes what the original function does.
2. We can find the inverse of a function by swopping all the ’s with ’s in the original function equation.
3. The inverse of a function e.g. is denoted by .
4. The domain of becomes the range of and the range of becomes the domain of .
5. The graphs of and are symmetrical about the line .
6. It is important to remember that the inverse of a function is not always a function itself, as we saw in the last activity.
7. Sometimes, we have to restrict the domain of the original function to make sure that its inverse is also a function.

Get some practice working with inverses of functions by doing the next activity.

### Activity 8: Inverses of Functions

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will give you practice working with the inverses of functions. |
| Stopwatch | Suggested Time You will need about 35 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

1. Given .
2. Write down the equation for .
3. Draw sketches of and on the same system of axes. Also draw the line of reflection, .
4. Write down the domain and range of and .
5. Is ?
6. Given
7. Write down the equation for .
8. Write down the domain and range of and .
9. Draw sketches of and on the same set of axes. Also draw the line of reflection, .
10. Is a function?
11. Given .
12. Write down the equation for .
13. Write down the domain and range of and .

#### Guided Reflection

1. .
2. To find the inverse of we need to write the equation as and swop all the ’s and ’s.

So

1. Here is the sketch of and .

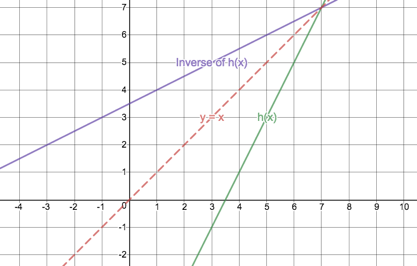


Image source: <https://www.desmos.com/calculator/2c5nxzqkh1>

1. Domain: Range:

Domain: Range:

1. is a function.
2. To find the inverse of we need to write the equation as and swop all the ’s and ’s.

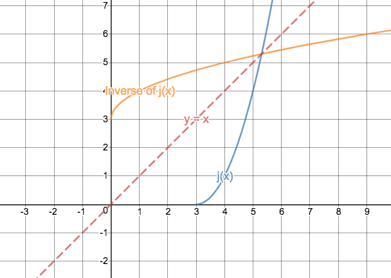
So but

(because , we know that we only need to worry about and can ignore )

1. Domain: Range:

Domain: Range:

1. Here are sketches of and . Notice how the original restriction of the domain of means that we only draw one arm of the parabola. This then also means that we only draw one arm of the inverse.



1. is a function.
2. .
3. To write down the equation of , we can follow the same steps as finding the inverse of a function. Start by writing the inverse as and then swop the variables.

So,

Domain: (remember that we can never of zero in a denominator)

Range: ( can never equal zero so can never equal )

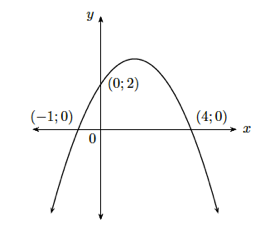
Domain: Range:

### Activity 9: Finding Equations of Quadratic Functions

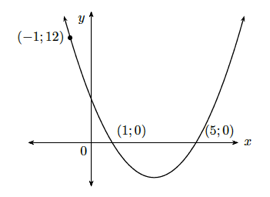
|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you consolidate your knowledge of quadratic functions by finding the equations of quadratic functions from their graphs. |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

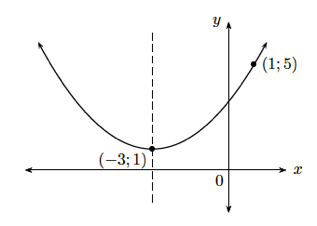
1. The following is a sketch of the function .



1. What kind of fuction is ?
2. If the x-intercepts are and , set up an equation in the form of .
3. Use the fact that the fact that the y-intercept is 2, to solve for in the equation above.
4. Write down the equation of .
5. Find the equation of the parabola below.



1. The followig graph is of the function .



1. Set up an equation of the form of to using the TP.
2. Use the point to solve for .
3. Write down the equation for .
4. A parabola has an AS of and a TP on the x-axis. It passes through the point . Find the equation of the parabola.
5. A parabola has a AS of and passes through the point . One of its x-intercepts is . Find the equation of the parabola.
6. shares a y-intercept with and has a TP of . Where else do and intersect?

#### Guided Reflection

### Activity 10: Applications of Quadratic Equations and Review

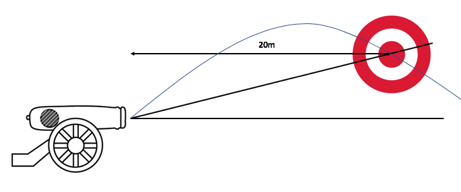
|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you review and consolidate your knowledge of the quadratic functions. |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

1. A farmer needs to divide up his square plot for each of his children. He decides to give his youngest child a plot of ground which was divide by decreasing the original plot on one side by 60m and the on the other sides by 80m. This plot is 2,400m2. Was were the original dimensions of the plot?
2. A warehouse has the shape of a rectangle 21m long and 10m wide. Determine the thickness of its walls if the inside area, is 180m2.
3. Given the function .
4. Write down the domain and range of .
5. Draw a sketch of .
6. Find the equation of .
7. Where do and intersect?
8. A target is a horizontal distance of 20m from a cannon but is on an inclined plane (like on the side of a hill) with a gradient of 0.15. The cannon fires a cannon ball up the plane such that the cannon ball’s path can be described by . Will the cannon ball hit the target?
9. A javelin thrower makes a throw. It lands 150m away and reaches a maximum height of 45m. If the javelin’s flight follows a parabola, work out the equation of this parabola. Assume that the path goes through the point .

#### Guided Reflection

1. .
2. .
3. .
4. Here is a diagram illustrating the situation.



The inclined plane can be described by a straight line with the equation . We also know that the flight of the cannon ball is given by . We need to see where these two graphs intersect by solving these equations simultaneously. If one of the points of intersection has an x-coordinate of 20, then we know that the target is hit.

(multiply through by 500)

or

We can ignore the negative root. Because the cannon ball will land a horizontal distance of 25m away from the cannon, it will not hit the target.

1. In

## Unit 4: Hyperbolic Functions

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Define what a hyperbolic function is;
2. Sketch hyperbolic functions of the form
3. Explain the effects on the shape of the graphs of hyperbolic functions of , and ;
4. Describe the various characteristics (e.g. domain and range) of graphs of hyperbolic functions;
5. Determine the equation of hyperbolic functions from their graphs; and
6. Interpret the graphs of hyperbolic functions to make arguments or predictions.

#### Introduction

We said when introducing the quadratic function that the graphs they make, parabolas, were widely used in satellite dishes and mirrors in telescopes.

The hyperbolic function, whose graph we call a **hyperbola**, is also widely used in lenses and mirrors because it also focuses light to a single point.

In this unit, we will explore the hyperbolic function in more detail and discover some very important properties that make it special.

### Activity 1: Introducing the Hyperbolic Function

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the basic features of the hyperbolic function. |
| Stopwatch | Suggested Time You will need about 75 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. A scientist was doing some experiments with helium gas. He measured the volume a gas occupied (in m3) under different pressures (in kPa), all the while making sure to keep the temperature of the gas constant at 25. Here are his results.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Pressure (kPa) | 50 | 100 | 150 | 200 | 250 | 300 | 400 |
| Volume (m3) | 320 | 160 | 106.67 | 80 | 64 | 53.3 | 40 |

* 1. Plot these points on a Cartesian Plane and join them with a smooth curve. Let pressure be .
  2. What happens to the volume as the pressure increases?
  3. What happens to the pressure as the volume increase?
  4. Can you write a Mathematical expression that relates the pressure and volume of this gas at this temperature?
  5. Is the graph symmetrical? About what line is the graph symmetrical?
  6. Is there a piece of the graph that is missing? What happens when is negative? Can you plot this missing piece of the graph?

1. Have a look at this expression: .
2. Complete the following table using this expression

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |

1. Plot these points and join them with a smooth curve.
2. What happens when ?
3. What happens to when the value of gets very large or very small?
4. Will the graph ever touch or cross either of the axes? Why or why not?
5. Why does the graph have two separate curves?
6. Is a function? If so, write it in function notation.
7. What are the domain and range of this function?
8. About which two lines is the graph symmetrical?
9. Here are three hyperbolic functions: , and .
10. If all of these functions are in the form , using your previous knowledge of the effect of , describe how these three functions will differ.
11. Use a table of values to sketch each of these functions on the same set of axes to test your prediction in a).
12. If an asymptote is a straight line that a graph gets ever closer to but never touches or crosses, what are the horizontal and vertical asymptotes of each of these functions? How do these asymptotes compare to the equation of each function? Can you write a general expression for the horizontal asymptote?
13. Write down the domain and range of each function. How do these relate to the equations of the functions? Can you write a general expression for the range?
14. About which lines is each function symmetrical. Can you write a general expression for these axes of symmetry based on the equation of the function?
15. Now visit the interactive simulation at <https://www.desmos.com/calculator/guac8xfeks> and change the value of to confirm and consolidate your understanding of the effect of in .
16. Here are three hyperbolic functions: , and .
17. If all of these functions are in the form , using your previous knowledge of the effect of , describe how these three functions will differ.
18. Use a table of values to sketch each of these functions on the same set of axes to test your prediction in a).
19. If an asymptote is a straight line that a graph gets ever closer to but never touches or crosses, what are the horizontal and vertical asymptotes of each of these functions? How do these asymptotes compare to the equation of each function? Can you write a general expression for the horizontal asymptote?
20. Write down the domain and range of each function. How do these relate to the equations of the functions? Can you write a general expression for the range?
21. About which lines is each function symmetrical. Can you write a general expression for these axes of symmetry based on the equation of the function?
22. Now visit the interactive simulation at <https://www.desmos.com/calculator/guac8xfeks> and change the value of to confirm and consolidate your understanding of the effect of in .
23. Using any method, draw a sketch of .
24. What is the domain and range of .
25. Use the interactive simulation at <https://www.desmos.com/calculator/guac8xfeks> to confirm that you sketch is correct.

#### Guided Reflection

1. The relationship between the volume and pressure of a gas in this question is called Boyle’s Law.
2. If we plot these points and join them with a smooth curve, we get a graph that looks like this.

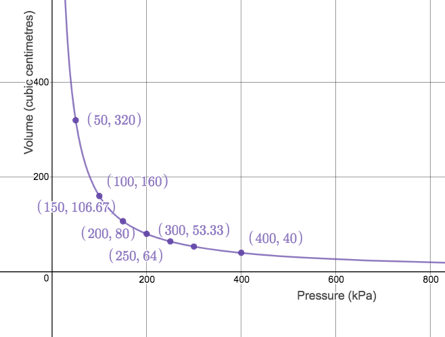
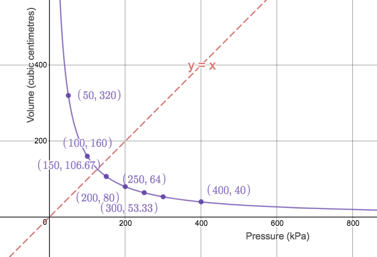
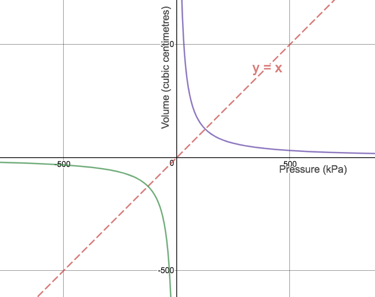


Image source: <https://www.desmos.com/calculator/te2jtk5nnt>

1. As the pressure increase, the volume gets smaller and smaller.
2. As the volume increases, the pressure gets smaller and smaller.
3. If we multiply the x- and y-coordinates of each point, the answer is always equal to 16,000. So, we can say that the expression is or .
4. The graph is symmetrical about the line . This makes perfect sense from what we know about the inverses of functions. If we reflect a graph about the line , we just swop the variables. If we do this to this graph, we get back the very same graph. We can write the relationship between and either as or .



1. Even through it makes no physical sense in terms of the pressure and temperate of a gas, if and we know that the point satisfies this equation, then the point will also satisfy the equation. This is true for each of the points we plotted. Here is the rest of the graph plotted.



1. We were given the expression .
2. Here is the completed table. Note that we don’t know what happens when .

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |
|  |  |  |  | ? |  |  |  |

1. Here is the graph of the expression.

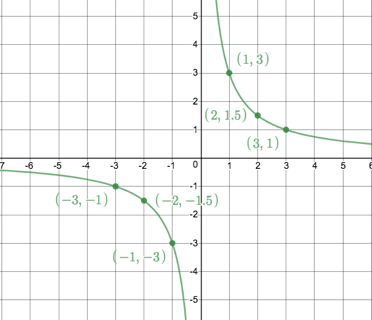
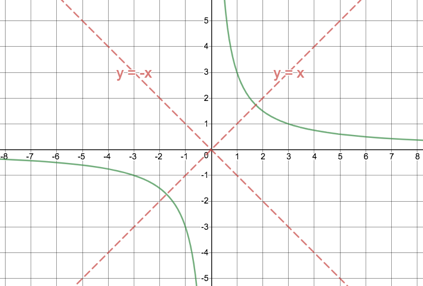


Image source: <https://www.desmos.com/calculator/5qsirdkvva>

1. We don’t know what happens when . The expression or is undefined because we are never allowed to divide by zero.
2. When gets very small, then the value of in gets very big. When gets very big, then the value of in gets very small. We can see this clearly in the sketch of the graph in both arms of the graph.
3. tells us that . Therefore, the graph will never touch or cross the y-axis. We can also right the expression as. This tells us that the graph will never touch or cut the x-axis either. In this case the x- and y-axes are called **asymptotes**. These are straight lines that a graph can get closer and closer to without ever touching or crossing.
4. There are two arms to the graph because a and a .
5. is a function. At no point will we ever get more than one output for any given input. We can check this with the vertical line test. We know that the graph will never touch or cross the y-axis so, even though it might look like it, does not give two outputs. In fact, , gives no output because it is undefined.
6. Domain: – can be any real number so long as it is not zero.

Range: – we can produce any real output except zero.

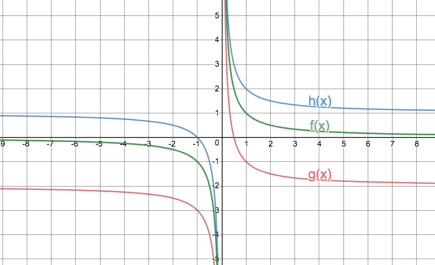
1. The graph is symmetrical about and .



1. Here are three hyperbolic functions: , and .
2. From and we expect the value to to move the whole graph vertically up or down. We expect that will be shifted 1 unit *up* from and that will be shifted 2 units *down* from .
3. Here is a completed table of values.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |
|  |  |  |  | ? |  |  |  |
|  |  |  |  | ? |  |  |  |
|  |  |  |  | ? |  |  |  |

Here are the three functions sketched. We can see that has been shifted 1 unit up and that has been shifted 2 units down.



1. The vertical asymptote of all three functions is the y-axis or the line . Each graph has a different horizontal asymptote.

horizontal asymptote is .

horizontal asymptote is .

horizontal asymptote is .

The value of each horizontal asymptote is the same as the value of . In general, the horizontal asymptote is the line .

1. Here is the domain and range of each function.

Domain: Range:

Domain: Range:

Domain: Range:

In each case, the domains are the same, but the ranges are different and reflect the position of the asymptote and the value of . In general, the range can be given as .

1. is symmetrical about the lines and .

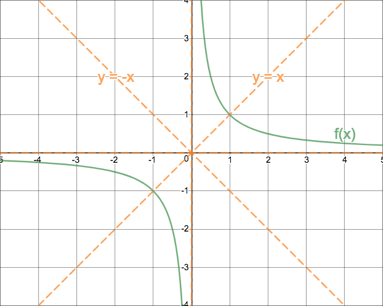
is symmetrical about the lines and .

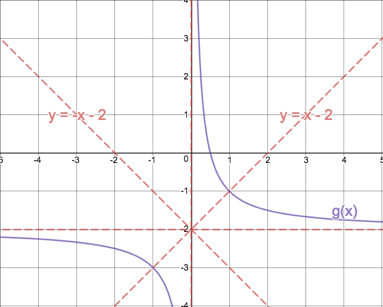
is symmetrical about the lines and .

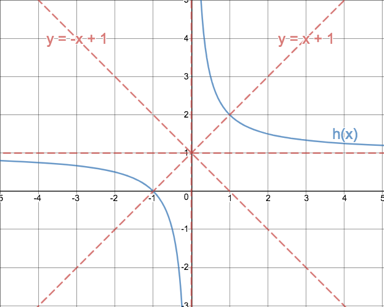
We can, therefore, write a general expression for the AS’s of these hyperbolic functions as

AS: and

1. Here are all three functions with their asymptotes and axes of symmetry.







1. Here are three hyperbolic functions: , and .
2. Based on and , we expect the value of to change the shape and orientation of the graph. Therefore, we expect to be wider than and to be flipped around.
3. Here is a completed table of values.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |
|  |  |  |  | ? |  |  |  |
|  |  |  |  | ? |  |  |  |
|  |  |  |  | ? |  |  |  |

Here are the three functions sketched. We can see that is now in the other two quarters of the Cartesian Plane (where the coordinates are always opposite in sign) and that is a wider, flatter graph than



1. The vertical asymptote of all three functions is the y-axis or the line . The horizontal asymptote of each graph is the x-axis. None of the graphs have been shifted up or down because in each case the value of was zero. Again, we can say that the horizontal asymptote is the line .
2. Here is the domain and range of each function.

Domain: Range:

Domain: Range:

Domain: Range:

In each case, the domains and ranges are the same. In general, the range can be given as but as in each case, this is .

1. is symmetrical about the lines and .

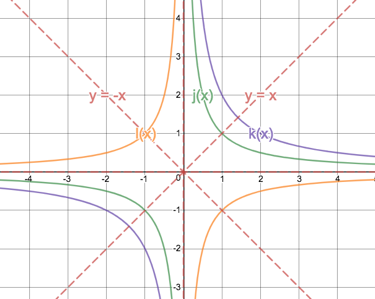
is symmetrical about the lines and .

is symmetrical about the lines and .

We can, therefore, write a general expression for the AS’s of these hyperbolic functions as

AS: and . For each of these functions, the value of was zero.

1. Here are all three functions shown with their asymptotes as well as their axes of symmetry.



1. To sketch the graph of we can make use of what we already know rather than a table of values. The value of is negative which means the graph exists in the two quarters (or **quadrants**) when the coordinates have different signs. The graph has also been shifted 1 unit up so there is a horizontal asymptote at . The vertical asymptote is still the line . We really just need one point on each arm. If we chose the points we get coordinates and . Our sketch looks like this.
2. Domain: Range:

Let’s summarise what we learnt in that last activity.

1. We know that the hyperbolic function can be written in the form .
2. When in this form, the value of tells us how much the graph has been sifted vertically up or down.
3. This means that the horizontal asymptote is given by .
4. This means that the range can be written as .
5. This means that the axes of symmetry are the lines and .
6. When in this form, we know that the domain is and the vertical asymptote is the line .
7. The value of tells us how flat and in what quadrants the graph exists.
8. If , then the graph is in the first and third quadrants (where the signs of the coordinates are always the same).
9. If , then the graph is in the second and fourth quadrants (where the signs of the coordinates are always opposite).
10. The bigger is, wider and flatter the graph is.

Here is a summary of these effects.

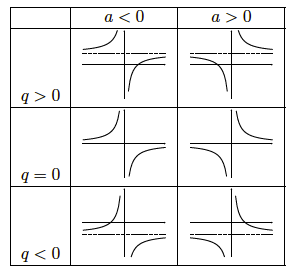


Image source: Everything Maths Gr10 pg170

We know the effects of and on the position and shape of the hyperbolic function. You are probably wondering what has happened to the general variable of like we had in the TP form of the quadratic function ? If, in both cases, is responsible for the shape and orientation of the graph and is responsible for vertical shifts, then maybe we can shift the hyperbolic function horizontally (left and right), if we have it in the form .

Let’s investigate.

### Activity 2: The General Form of the Hyperbolic Function

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will consolidate your understanding of the hyperbolic function by examining the general form |
| Stopwatch | Suggested Time You will need about 30 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. Look back at the functions we explored in Activity 1. If all of them were rewritten in the general form of , what was the value of in each case. How does this value relate to the graphs’ vertical asymptotes, domain and axes of symmetry?
2. What do you think the graph of looks like?
3. In which quadrants with the graph exist?
4. What will the horizontal asymptote be?
5. What will the vertical asymptote be?
6. What will the axes of symmetry be?
7. Find any two points on the graph.
8. Make a quick sketch of .
9. Visit the interactive simulation at <https://www.desmos.com/calculator/f4dskzapvy> and check that your sketch from question 2) was correct.
10. In which quadrants does the graph exist?
11. What is the horizontal asymptote?
12. What is the vertical asymptote?
13. What are the axes of symmetry?
14. Write a general expression for the axes of symmetry in terms of and . Hint: What do you notice about the values of and and the y-intercepts of the axes of symmetry?
15. Where do the axes of symmetry intersect and how does this point relate to and .
16. What are the domain and range?
17. Write a general expression for the domain and range in terms of and .
18. Make a sketch of drawing in the asymptotes and the axes of symmetry. You can check that your sketch is correct using the interactive simulation at <https://www.desmos.com/calculator/f4dskzapvy>.

#### Guided Reflection

1. If all the functions in Activity 1 were written in the form , the value of in all cases would be zero. In all cases we saw that the vertical asymptote of each function was the line and that the domain of each function was . So perhaps, if the value of , these would change to be and respectively.

We also saw that the axes of symmetry in all cases were the lines and . Because we think that the value of determines a function’s horizontal shift, the equations of these axes of symmetry will change but it is not clear yet how.

1. Because the value of , the graph will be in the first and third quadrants. Also, so the graph will be a bit flatter than one with the equation .
2. The horizontal asymptote will be the line . This means the graph is shifted 2 units *down*.
3. The vertical asymptote might be the line because the function can be rewritten as. This means the graph is shifted 1 unit to the *left*.
4. The one AS (without the horizontal shift) would be . But we think the whole graph will be shifted 1 unit to the left. This would increase the y-intercept of the line by 1 unit. So, we think the one AS will be .

The other AS (without the horizontal shift) would be . But we think the whole graph will be shifted 1 unit to the left. This would decrease the y-intercept of the line by 1 unit. So, we think the one AS will be .

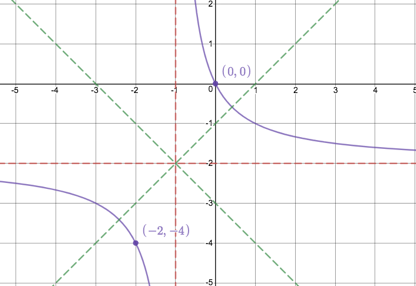
1. To find any two points on the graph you can try and find the intercepts.

x-intercept (let :

So, the graph goes through the origin . Therefore, the y-intercept is also zero. We need to find one other point. Ideally, this point should be on the other arm of the graph i.e. on the other side of the vertical asymptote. We can choose any convenient value for , say .

Therefore, the point also lies on the graph.

1. Here is the graph of .



1. The graph exits in quadrants one and three as expected for .
2. The horizontal asymptote is the line in keeping with the general rule of .
3. The vertical asymptote is the line as we expected where in and the equation of the vertical asymptote being .
4. The axes of symmetry are the lines and .
5. We know that the axes of symmetry are the lines and . They have y-intercepts of and respectively, which are one greater and one less than the value of . But the value of . Therefore, we can write the equations for the axes of symmetry as follows:

and or more generally as

and

1. The axes of symmetry intersect at the point . This is the point .
2. Domain: Range:
3. General expressions for the domain and range are:

Domain: Range:

1. We need to sketch .

: the graph exists in quadrant two and four.

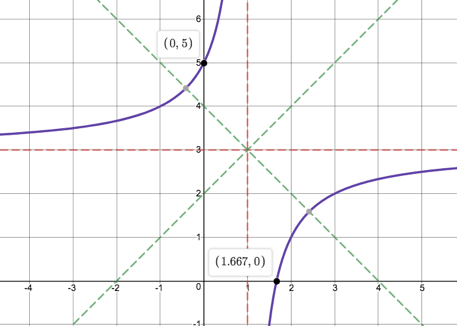
: the graph is shifted 3 units *up*. Therefore, the horizontal asymptote is .

: the graph is shifted 1 unit to the *right*. Therefore, the vertical asymptote is .

The axes of symmetry are and or and . As a second check, the axes of symmetry always intersect at the point .

x-intercept (let : y-intercept (let :

Here is a sketch of . Notice again, how the axes of symmetry intersect at the point .



### Activity 3: Finding the Equations of Hyperbolas

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to find the equations of hyperbolic functions from their graphs as well as solve other problems involving hyperbolas. |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. A hyperbola of the form pases through the point . If the axes of symmetry intersect at , determine the equation of .
2. Find the equation of the hyperbolic function represented in this graph.

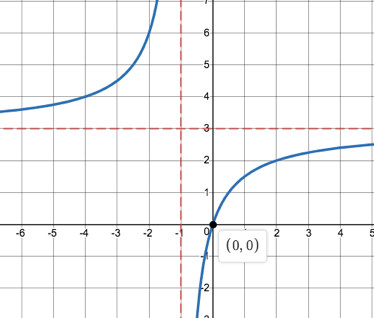


Image source: <https://www.desmos.com/calculator/16vokigzgl>

1. A hyperbolic function, , has axes of symmetry of and and a y-intercept of .
2. Find the equation of .
3. What is the equation of the straight line, , intersecting with and the points where and ?
4. What is the average gradient of between the points of intersection of and ? Illustrate this with a graph.
5. Is the gradient of at greater than or less than this average gradient? Illustrate this with a graph.

#### Guided Reflection

1. We are told that and that the axes of symmetry intersect at . That means that and . Therefore

We are also told that the graph passes through so we have values for and to substitute into the equation of to find .

1. We can see from the graph that the horizontal asymptote is the line . That means that . We can also see that the vertical asymptote is the line . Therefore, .

So, the equation of the hyperbola is of the form .

We can also see that the graph passes through the point . If we substitute these values for and into the equation, we get

Therefore, the equation of the function is . The negative value of makes sense given that the graph exists in quadrant two and four.

1. We are told that , has axes of symmetry of and and a y-intercept of .
2. We know that the point where the axes of symmetry meet is the point . To find where the axes of symmetry intersect, we need to solve their equations simultaneously.

(1)

(2)

Substitute into (1):

The axes of symmetry intersect at . Therefore and .

So, . Substitute the point (the y-intercept) into the equation to find .

Therefore .

1. We are given the x-coordinates of two points of intersection. We need to use to first find the y-coordinates.

. So, the point is .

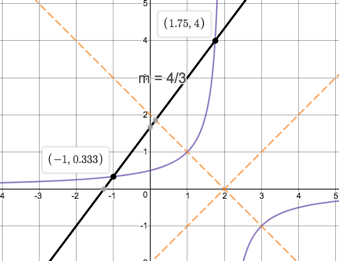
. So, the point is .

Now we can find the gradient of the straight line between these two points using .

The gradient of the straight line is . Therefore, it has an equation . We can substitute either of out points into this equation to find Let’s use .

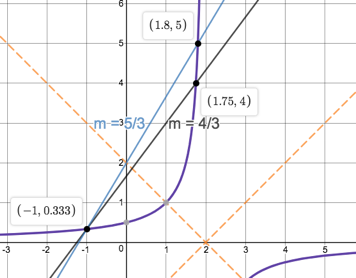
The equation of the straight line is .

1. If the gradient of the straight line between these points of intersection is , then it means that the average gradient of the hyperbola between these points is also . Here is a graph showing what this means.



1. The gradient of is greater than the average gradient between the points and . In order for a line to pass through and this new point , it would need to have a gradient of . If the average gradient between these two points is greater than the average gradient between the original two points, it must mean that the gradient at is greater than the original average gradient.

Here is the situation illustrated graphically.



The concept of average gradient we encountered in the previous activity is a new one. In simple terms,

**the average gradient between any two points on any graph is simply the gradient of the straight line between these two points.**

Let’s explore average gradient a little more.

### Activity 4: Average Gradient

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you explore and formalise the concept of average gradient of a curve. |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

1. Given :
2. Draw a sketch of .
3. Determine the average gradient of between the points where and .
4. Given the curves and :
5. What is the average gradient of between and ?
6. What is the average gradient of between and ?
7. Are and increasing or decreasing between these points?
8. Between which point and the point is the average gradient of equal to ?
9. What is the equation of this average gradient straight line, ?
10. Sketch and on the same set of axes?
11. What point is half-way between the points of intersection of and ? What do you think we can say about the gradient of at this point?

#### Guided Reflection

1. We are given .
2. Firstly, is a quadratic function. We know from the value of that the graph has a minimum TP. We also know that , and .

: The graph has a y-intercept at .

We can work out the AS using .

Because the graph is symmetrical about the y=axis, we know that the TP and y-intercept are the same point.

x-intercepts (let ): .

Find one other point for shape: .

Here is the sketch.

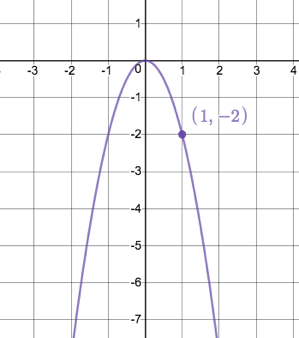
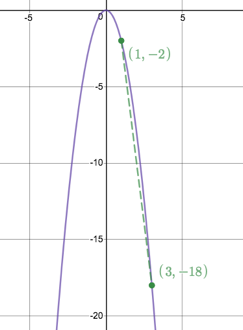


Image source: <https://www.desmos.com/calculator/edgbhgsdig>

1. We know that the average gradient of the curve between any two points is the same as the gradient of the straight line between those to points. We are asked to find the average gradient between and .



1. We are given the curves and .
2. The average gradient between and of is given by
3. The average gradient between and of is given by
4. The average gradient of is *positive* which means that the function is *increasing* – as gets bigger, gets bigger.

The average gradient of is *negative* which means that the function is *decreasing*- as gets bigger, gets smaller.

1. We know we are looking for an average gradient of . Therefore, we know that

We also know that one of the points is the point .

So,

or

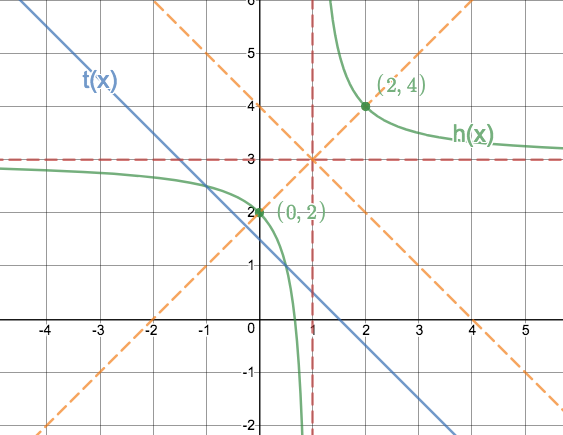
The point where is the one we already know of. Therefore, we need to find the other point where .

So, the other point is

1. We know that the gradient of is 1. Therefore . We can substitute either point we have on the graph to find the value of . Let’s use .

Therefore

1. Here are the sketches of and .



1. The point is half-way between the points of intersection of and ? Because this point is half-way and because is symmetrical about the line and this point lies on , we can say that the gradient at this point is equal to the average gradient between the points of intersection of and i.e. the gradient at .

## Unit 5: Exponential Functions

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Define what an exponential function is;
2. Sketch exponential functions of the form ;
3. Explain the effects on the shape of the graphs of exponential functions of , , and ;
4. Describe the various characteristics (e.g. domain and range) of graphs of exponential functions;
5. Determine the equation of exponential functions from their graphs; and
6. Interpret the graphs of exponential functions to make arguments or predictions.

Introduction

You have probably heard the term “exponential growth” before. Often, however, the word “exponential” is used incorrectly to just refer to something that happens very quickly. In Mathematics, exponential has a very specific meaning and exponential functions are specific kinds of functions.

The good news is that these functions behave quite similarly to the other functions we have explored so most of what we know about linear, quadratic and hyperbolic functions can be applied to exponential functions as well.

There are many processes in nature and finance where the relationship between the inputs and outputs is exponential. One example is how bacteria reproduce. Bacteria reproduce by simply splitting into two new bacteria. There is no sex involved.

So, if one bacterium splits into two and then each of these splits into two and then each of these splits into two etc. pretty soon we have a whole lot of bacteria!

### Activity 1: Introducing the Exponential Function

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the basic features of the exponential function. |
| Stopwatch | Suggested Time You will need about 75 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An calculator |

#### Tasks

1. There is a story about the inventor of chess. When he presented the game to the king, the king was so impressed, we wanted to reward the inventor. The inventor asked for rice. He asked that one grain of rice be placed on the first square of the board and that the number of grains on each successive square be double the previous square. So, the second square would have 2 grains, the third square 4 grains, the fourth square 8 grains and so on.

The king thought the inventor silly for asking for such a small reward and so agreed.

* 1. How many grains of rice would there be on the 4th, 5th and 6th squares?
  2. Complete this table for the number of squares and the number of grains of rice.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Square |  |  |  |  |  |  |
| Grains of rice |  |  |  |  |  |  |

* 1. Plot these points and join them with a smooth curve.
  2. Write an expression that describes the relationship between the inputs and outputs. Hint: the outputs increase in powers of 2. Look at the value of needed to give each output and compare this with the power of 2 that the output is.
  3. If there are 64 squares on a chess board, how many grains of rice would be on the last square?
  4. What happens when ?
  5. What happens when ? Try a few values of to find out.
  6. Does the graph have a horizontal asymptote? If so, what is this asymptote?
  7. Does the graph have a vertical asymptote? If so, what is this asymptote?
  8. What is the domain and range of this graph?
  9. Complete the sketch of your graph using this new information.
  10. Is the relationship between and a function? Why?

1. Imagine for a moment that the inventor of chess asked that the number of grains of rice on each successive square of the board be tripled instead of doubled.
2. Complete this table of values for this case.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Square |  |  |  |  |  |  |
| Grains of rice |  |  |  |  |  |  |

1. Write an expression for this relationship. Is it a function?
2. Sketch this function on the same set of axes as question 1).
3. What is the domain and range of this function?
4. What is the same about this function and the one from question 1)?
5. What is different about this function and the one from question 1)?

#### Guided Reflection

1. We are dealing with a relationship between the number of the chess board square and the number of grains of rice on each square.
2. On the first square there is 1 grain.

One the second square there are 2 grains.

On the third square there are 4 grains.

On the fourth square there are 8 grains.

On the fifth square there are 16 grains.

On the sixth square there are 32 grains.

1. Here is the completed table.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Square |  |  |  |  |  |  |
| Grains of rice |  |  |  |  |  |  |

1. Here are the points in the table plotted and joined with a smooth curve.

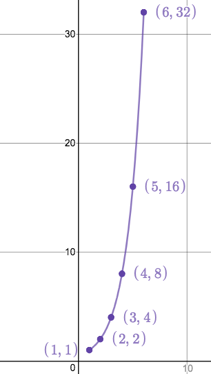


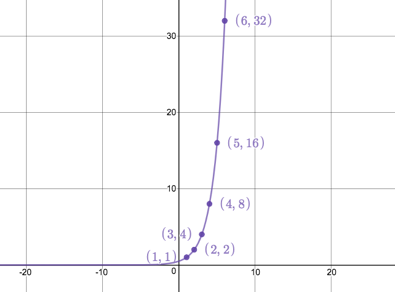
Image source: <https://www.desmos.com/calculator/zbgupglpal>

1. Here is the table of values re-written with the output values as powers of 2.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Square |  |  |  |  |  |  |
| Grains of rice |  |  |  |  |  |  |

Each time, the exponent on the base is one less than the input value so the expression is

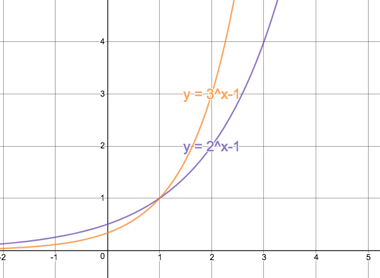
1. The number of grains on the 64th square will be . This is more than 9 million billion – a very big number!
2. When , .
3. When , the y-values are fractions because the base is raised to a negative power.
4. No matter how negative we let , the y-values will still always be positive. Therefore, the x-axis or line is a horizontal asymptote.
5. The graph rises very steeply but there is nothing stopping getting bigger and bigger. Also, there is nothing stopping getting more and more negative either. Therefore, there is no vertical asymptote.
6. Domain: Range:
7. Here is a sketch of the whole graph.



1. This relationship is a function. For every input, we only ever get a single output. A vertical line will never cut the graph more than once.
2. Now the number of grains of rice on each successive square is tripled.
3. Here is the completed table of values for this case.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Square |  |  |  |  |  |  |
| Grains of rice |  |  |  |  |  |  |

1. This expression is the same as the one in question 1) except that we are now dealing with powers of three. Therefore Like the other relation, this one is also a function.
2. Here is a sketch of both functions but only for values near the origin.



1. Domain: Range:

This is exactly the same as the other function.

1. We have just seen that the domain and range are the same. The line is also a horizontal asymptote for both functions. Both graphs go through the point . This makes sense as at both points the exponent is zero and any base to the power of zero is defined as .
2. The function rises more steeply than . It also has a y-intercept of instead of . It also approaches its horizontal asymptote more quickly.

So far, we know the general shape that exponential functions have. They increase very rapidly as the values of get bigger and they approach a horizontal asymptote as the values of get smaller and smaller.

Let’s investigate exponential functions some more.

### Activity 2: Exponential Functions of the Form

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the basic features of the exponential function. |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. Visit the interactive simulation at <https://www.desmos.com/calculator/u0itwiodvc>. Here you will find the exponential function .
2. If the general form of this function is , What are the current values of , and ?
3. Based on your knowledge of the effect of in , and , what do you think the graphs of the functions and will look like. Use the interactive simulation to test your hypothesis and then sketch , , and on the same set of axes making sure that you include the horizontal asymptote, y-intercept and one other point of each function.
4. Write down the domain and range of each function.
5. Write general expressions for the domain, range and the horizontal asymptote in terms of .
6. Are all these functions increasing? Why do you say so?
7. Given , and .
8. How will the graphs of , and differ? Create a table of values like this one to prove your prediction.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

1. Draw sketches of these three functions showing the horizontal asymptotes, y-intercepts and one other point.
2. Write down the domain and range of each function.

#### Guided Reflection

1. We are dealing with exponential functions of the form .
2. The current values are , and .
3. . Therefore, . Based on how affects other functions, we expect to be shifted 1 unit *up*.

. Therefore, . Based on how affects other functions, we expect to be shifted 2 units *down*.

Here is a sketch of all three functions.

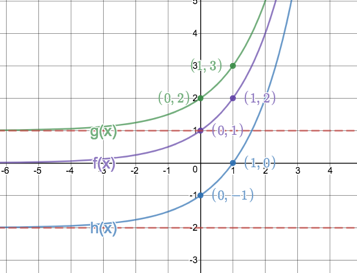


Image source: <https://www.desmos.com/calculator/f0owx8bf7c>

1. : Domain: Range:

: Domain: Range:

: Domain: Range:

1. The domain is not affected by the value of and so is always .

The range depends on the value of such that the range is .

The horizontal asymptote is given by .

This is all exactly the same as with hyperbolic functions.

1. Each of these functions is increasing. As gets bigger, gets bigger.
2. We have the following functions:

()

(

()

1. Based on our knowledge of how affects other functions, we expect to have a wider shape than . We expect the graph of to be narrower than and flipped upside down because it is negative.

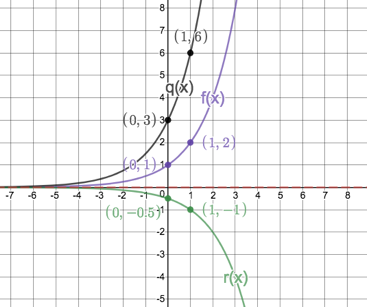
Here is the completed table of values.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

We can see from the table of values that performs a transformation of the function. For , each output value is i.e. . For , each output value is i.e. .

This means that will be a stretched-out version of and will be a squashed and inverted version of .

1. Here are sketches of all three functions showing these transformations. Notice how all three functions have the same horizontal axis of symmetry because none of them have been shifted up or down. In all cases .



1. : Domain: Range:

: Domain: Range:

: Domain: Range:

Notice that, because is inverted, its range is also inverted i.e. .

From Activity 1, we already have a sense of how the value of affects the shape of the graph. But let’s investigate further because things may not be as simple as we think.

### Activity 3: What does do in ?

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will give you insight into the effect that changing the value of has on the exponential function and why we need to restrict what values have to make sure that the function is well behaved. |
| Stopwatch | Suggested Time You will need about 50 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. Visit the interactive simulation at <https://www.desmos.com/calculator/h40txbfcnz>. Here you will find three functions, , and as well as the general function , with the , and with a slider to change the value of .
2. What is the y-intercept of , and ? Why is this? What can we do to change the y-intercept of an exponential function?
3. What happens to the exponential function when ? Why? What is the domain and range of the function?
4. What happens to the exponential function when ? Why? What is the domain and range of the function?
5. What happens to the exponential function when ? Think about the cases when and . Create a table of values of each of these cases. When is actually still a function? Can we represent the function as a single continuous curve like the other functions we have been dealing with?
6. Draw in neat little sketches in each block to indicate the general position and orientation of the exponential function in each case.

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

1. What is the equation of the function which is a reflection of about the y-axis? Use the general function and the slider for to experiment if you need to.
2. What is the equation of the function which is a reflection of about the x-axis? Use the general function and the slider for to experiment if you need to.
3. What is the equation of the inverse of ? Is this relation a function? Use the general function and the slider for to experiment if you need to.

#### Guided Reflection

1. We are working with the functions , and .
2. The y-intercept of all three functions is the point . This is because any base raised to the power of zero is equal to 1. Therefore, it does not matter what the value of is in . The output value when the input value is zero will always be 1.

In order to change this y-intercept, we can either shift the graph up or down by making the value of or we can transform the graph by making the value of .

1. When , the value of is always equal to 1 irrespective of the value of . One raised to any power is always equal to one. Therefore, if the value of and , is simply the straight line .

The domain can be any real number, but the range is restricted to a single output value. So, the domain and range of are:

Domain: Range:

1. When , the value of is always equal to 0 irrespective of the value of . Zero raised to any power is always equal to zero. Therefore, if the value of and , is simply the straight line .

But, the domain of the function also has to be restricted. We cannot allow any values of . This is because, by definition, and we can never have a zero denominator and the fact that is also undefined.

The range is also restricted to the single value of zero. So, the domain and range of are:

Domain: Range:

1. If or we have these values:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
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From this table, we can see that the output values jump around and alternate from positive to negative. This makes sense, because a negative base to an *even* power will be *positive* but the same negative base to an *odd* power will be *negative*.

Therefore, the function is not a smooth curve.

Also, the domain has to be restricted. Take for example. By definition, this is the same as and we cannot take the square. Any value of that would need us to take the square (or fourth or sixth or any even) root cannot be part of the domain. So, when , is not a solid curve and is discontinuous. However, it is still a function because we will only ever get a single output for every input.

1. Here is the completed table showing the general shape and position of in each case. We can see that when , the graph lies *above* the horizontal asymptote and when , the graph lies *below* the horizontal asymptote.

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  | Above the asymptote  The function is *increasing* | Below the asymptote  The function is *decreasing* |
|  | Above the asymptote  The function is *decreasing* | Below the asymptote  The function is *increasing* |

1. We know that when we reflect a function about the y-axis, we basically have to replace every with a . Remember the reflected function of about the y-axis will be .

If we are reflecting about the y-axis, then the reflected function will have the equation or . If we set the slider for , we can see that the created function is a mirror image of about the y-axis.

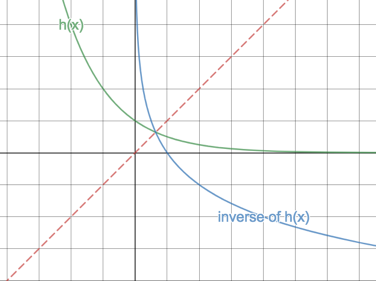
1. We know that when we reflect a function about the x-axis, we basically have to replace every with a . Remember the reflected function of about the x-axis will be .

If we are reflecting about the x-axis, then the reflected function will have the equation or . If we set the slider for and the slider for , we can see that the created function is a mirror image of about the x-axis.

1. We know that to find the inverse of a function we have to swop the variables around. Therefore, to find the inverse of :

. So, the inverse is . We know from our work in Topic 1 on logarithms that we can rewrite this as .

Here is a sketch of and its inverse showing that the inverse is also a function.



We saw in question 3) that for certain values of the exponential function is very badly behaved. When dealing with the exponential function, we almost always place restrictions on the values of to make sure that the function is well behaved. We say that

, for and

We also saw that the inverse function of the exponential function is the logarithmic function.

So far, we have dealt with the effects of , and on the function . Based on your knowledge of the other functions we have studied, you should be able to predict the effect of in . Let’s see if you are right.

### Activity 4: Sketching Exponential Functions

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will consolidate your understanding of the exponential function in order to be able to sketch their graphs. |
| Stopwatch | Suggested Time You will need about 50 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

1. Here are two exponential functions: , and .
2. Complete this table.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
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1. Plot each function on the same set of axes.
2. What is the domain and range of each function?
3. What is the horizontal asymptote of each function?
4. What is the y-intercept of each function?
5. What is the effect of on the exponential function ?
6. Give the domain and range of each of the following functions:
7. Sketch the following functions by determining the intercepts, horizontal asymptote, domain and range and general shape and position.

#### Guided Reflection

1. We were given the functions , and .
2. Here is the completed table of values.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
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1. Here is a sketch of all three functions.

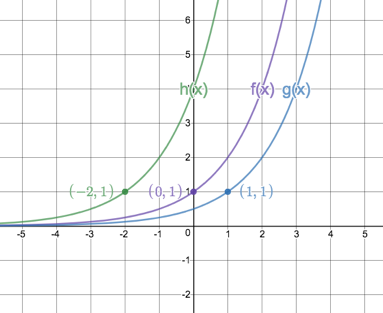


Image source: <https://www.desmos.com/calculator/ho3zf3yagg>

1. : Domain: Range:

: Domain: Range:

: Domain: Range:

1. Each function has the same horizontal asymptote, the line .
2. : y-intercept is the point .

: y-intercept is the point .

: y-intercept is the point .

1. shifts the graph horizontally left and right. The graph of (where ) is shifted 1 unit to the *right*. The graph of (where ) is shifted 2 units to the *left*.
2. We needed to find the domain and range of each function.
3. Given . This is an exponential function where the value of . Therefore, there is no vertical shift. , so, the graph exists above the horizontal asymptote.

Domain: Range:

1. Given . This can be rewritten as . so, the graph exists above the asymptote but so there is a vertical shift of units.

Domain: Range:

1. Given . This can be rewritten as . so, the graph exists below the asymptote and so there is a vertical shift of up of units.

Domain: Range:

1. Given . This can be rewritten as . so, the graph exists above the asymptote and so there is a vertical shift of down of units.

Domain: Range:

1. Given . This is not an exponential function. It is a quadratic. so, the TP is a minimum. We can either complete the square and get the function into the TP form to find the TP or we can use AS: to find the AS and then use this value to find the y-coordinate of the TP. Both methods are shown.

**TP form:**

Therefore, the TP is the point

Domain: Range:

**AS:**

Domain: Range:

1. We need to sketch the given functions.
2. Given .

x-intercept (let ): y-intercept (let ):

. Therefore, horizontal asymptote is

. Therefore, graph exists above the asymptote.

. Therefore, graph is increasing.

Domain: Range:

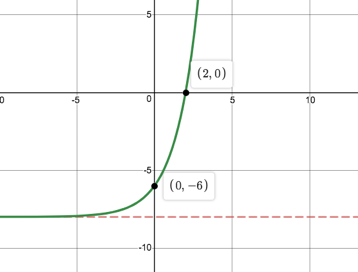


Image source: <https://www.desmos.com/calculator/3oeuox8ns5>

1. Given . Therefore

x-intercept (let ): y-intercept (let ):

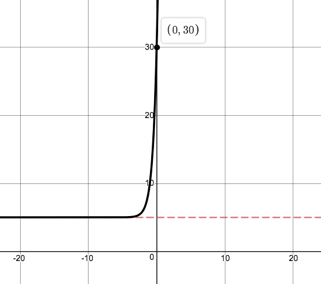
No solution

. Therefore, horizontal asymptote is

. Therefore, graph exists above the asymptote (hence no x-intercepts)

. Therefore, graph is increasing.

Domain: Range:



1. Given

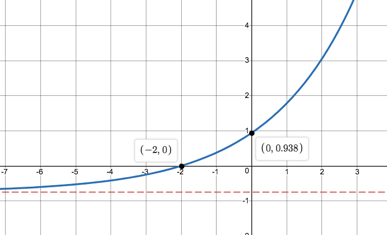
x-intercept (let ): y-intercept (let ):

. Therefore, horizontal asymptote is .

. Therefore, graph exists above the asymptote

. Therefore, graph is decreasing.

Domain: Range:



### Activity 5: Finding the Equations of Exponential Functions

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to find the equations of exponential functions from graphs or other information. |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

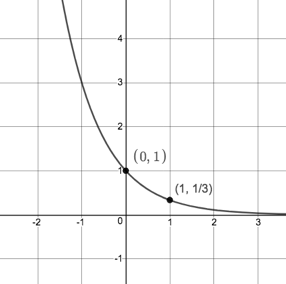
#### Tasks

1. If , use the graph to find the values of and .

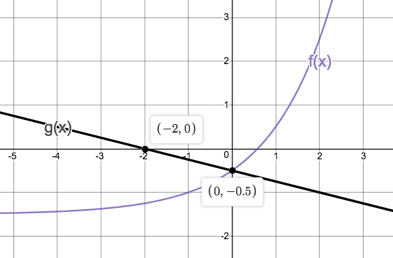


Image source: <https://www.desmos.com/calculator/jh4ojbqkbt>

1. Given and its graph, answer the following questions.



1. Show that .
2. Find the value of if the point lies on .
3. Calculate the average gradient of between and .
4. Determine the equation of the new function if Is shifted 2 nits down and 2 units to the left.
5. Find the equations of and if .



#### Guided Reflection

TBC

### Activity 6: All Together Now

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to consolidate all the knowledge you have gained about functions so far and to start learning how to interpret functions. |
| Stopwatch | Suggested Time You will need about 60 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

Everything Maths Ex5-19 pg193 Q1, 2, 3, 4, 6a, 6c, 9

## Unit 6: Trigonometric Functions

#### Learning Outcomes

By the end of this unit, you should be able to:

1. Define what the sine, cosine and tangent functions are;
2. Sketch sine functions of the form , and ;
3. Sketch cosine functions of the form , and
4. Sketch tangent functions of the form , and
5. Explain the effects on the shape of the graphs of trigonometric functions of , , and ;
6. Describe the various characteristics (e.g. domain and range) of graphs of trigonometric functions;
7. Determine the equation of trigonometric functions from their graphs; and
8. Interpret the graphs of trigonometric functions to make arguments or predictions.

Introduction

All modern music is produced with the aid of computers at some stage in the recording and mastering process. Computers are used to manipulate, transform and blend the sounds made by instruments of various kinds. These computer programmes use trigonometric functions to help them model and change sounds.

When sound travels through the air, the energy of the sound creates pockets of air molecules at slightly different pressures. In some of these areas the air is slightly compressed. In others, it is at a lower pressure.

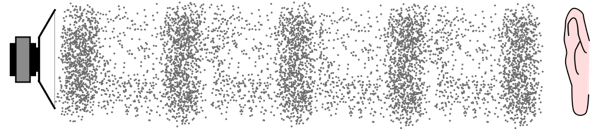
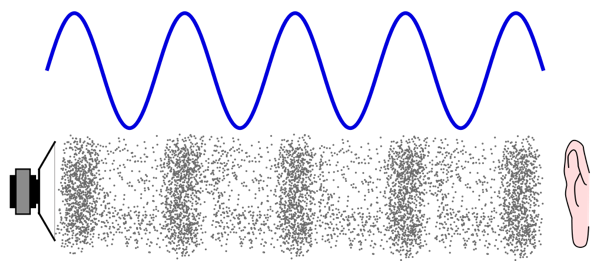


Image source: <https://commons.wikimedia.org/wiki/File:CPT-sound-physical-manifestation.svg>

These sound waves can be thought of in the same way as waves travelling across a smooth pond, where the areas of compression corresponding to the crests and the areas of low pressure correspond to the troughs.



The “water” waves can be modelled mathematically using the trigonometric function called sine. If we can model the wave using a function, it means we can manipulate it in all sorts of ways. Computers, being particularly good at maths, can make very complicated changes to very complex waves quickly and easily.

This is just one example of where trigonometric functions are used in real life. Other examples include anytime an engineer or architect needs to calculate forces or angles of forces that will be experienced by physical structures like buildings or bridges.

In this unit, we will not cover the basic theory behind the trigonometric functions of sine, cosine and tangent. If you need to learn about this or would like to revise it, please visit [reference] first.

### Activity 1: The Sine Function

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to one of the most widely used of the three trig functions – the sine function. |
| Stopwatch | Suggested Time You will need about 40 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * A calculator |

#### Tasks

1. Draw a graph of .
   1. Make a copy of this table and use your calculator to calculate the values for . If you need help using your calculator watch the video called Calculating Trig Ratios on your Calculator (mm:ss).

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
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* 1. Plot each of the points on a Cartesian plane and join them with a smooth curve.
  2. In order to get the smoothness of the curve correct, work out additional points by working out the value of for angles between the angles given in the table above.
  3. What is the greatest value of ? What is the smallest value of . What are the turning points of ?
  4. What do you notice about the value of when and when ? What do you think will happen to the graph if we calculate for angles bigger than What does an angle greater than 36 mean?
  5. What do you think will happen to the graph if we calculate for angles less than ? What does an angle less than mean?
  6. Is a function? Why?

1. Draw a graph of .
2. Copy and complete this table. Again, use your calculator to calculate the value of .

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |
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1. Plot each of the points on a new Cartesian plane and join them with a smooth curve. In order to get the smoothness of the curve correct, work out additional points by working out the value of for angles between the angles given in the table above.
2. What is the greatest value of ? What is the smallest value of . What are the turning points of ?
3. What do you notice about the value of when and when ? What do you think will happen to the graph if we calculate for angles bigger than
4. What do you think will happen to the graph if we calculate for angles less than ? What does an angle less than mean?
5. Is a function? Why?
6. Draw over your graph of with a thick black pen. Place your graph of over this graph so that the graph of is visible through the paper.
7. How is the graph of the same as ?
8. How is the graph of different from ?
9. Write an equation that describes the relationship between and .

#### Guided Reflection

1. We need to draw a graph of .
2. Here is the completed table of values.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |
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1. This is what your graph may have looked like without any additional points



Image source: ASECA Unit 2 (pg 60)

1. This is what the smoother curve looks like with some additional points.

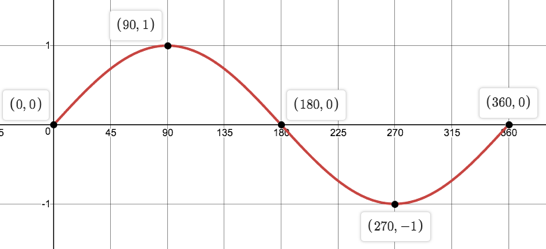


Image source: <https://www.desmos.com/calculator/tarywidnyv>

1. The graph of is never greater than 1. The point is its maximum. The graph is also never less than . The point is its minimum. Therefore, has two turning points – a maximum TP at and a minimum TP at .
2. The value of when and when are the same. They are both zero. If we calculate the value for values of the graph will repeat the same pattern as it did between and . An angle greater than means that we have gone through one full revolution and are back where we started and starting another revolution.

For example, take the angle . This is the same angle as except that we have already made one full revolution of . .

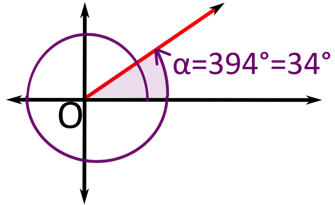


Image source: <https://upload.wikimedia.org/wikipedia/commons/thumb/1/18/Angle_bigger_360.svg/2000px-Angle_bigger_360.svg.png>

1. The graph will repeat the same pattern as it did between and . An angle less than means that we are turning in the *opposite* direction as before. So, if positive angles mean we turn in a counter-clockwise direction, negative angles are measured in a clockwise direction.

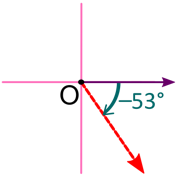
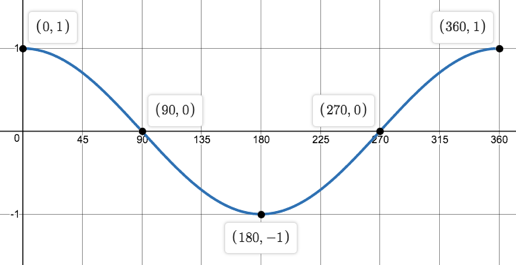


Image source: <https://upload.wikimedia.org/wikipedia/commons/thumb/6/63/Angles_negative_big.svg/1280px-Angles_negative_big.svg.png>

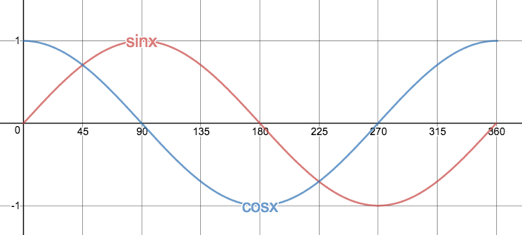
1. is a function. For every value of (input) there is one and only one value of output.
2. We need to draw a graph of .
3. Here is a completed table of values.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |
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1. This is what the smoother curve looks like with some additional points.



1. The graph of looks like it is never greater than 1. The points and are its maximum. The graph is also never less than . The point is its minimum. Therefore, has maximum turning points at and and a minimum TP at .
2. The value of when and when are the same. They are both one. If we calculate the value for values of the graph will repeat the same pattern as it did between and . An angle greater than means that we have gone through one full revolution and are back where we started and starting another revolution.
3. The graph will repeat the same pattern as it did between and . An angle less than means that we are turning in the *opposite* direction as before. So, if positive angles mean we turn in a counter-clockwise direction, negative angles are measured in a clockwise direction.
4. is a function. For every value of (input) there is one and only one value of output.
5. We need to compare the graphs of and . Here are both graphs on the same set of axes.



1. Both graphs are in the shape of waves. Both graphs have a maximum value of and a minimum value of . Both graphs seem to repeat themselves for values of and .
2. Even though both graphs have the same y-values of their turning points, the x-values of the turning points are different. It looks like the graph of is just the graph of shifted to the left. starts at zero but starts at one.
3. If is the same as shifted to the left, then we can say that or that .

From this last activity we have seen that and are both functions and are both very similar to each other. In fact, we saw that the cosine function is the same as the sine function except that it has been shifted to the left.

We also saw that the graphs seem to repeat themselves for angles that are outside the domain of {}.

Let’s double check that this is the case and investigate the sine and cosine functions a bit more.

### Activity 2: Periods and Amplitudes

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to understand what we mean by the period and amplitude of the sine and cosine functions. |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. Visit the interactive simulation at <https://www.desmos.com/calculator/xuivyb2swn>. Here you will find a circle with a radius of 1 and a slider with which you can change the size of angle from to . At the moment, the angle is at .
2. If is one full turn around, the circle, how many turns can you make between and ?
3. Increase the angle from to . How many revolutions or turns around the circle is this? How many repeats of the sine function pattern has the graph made?
4. Now continue to increase the angle to . How many complete revolutions around the circle have you made in total so far? How many patterns has the graph repeated?
5. Increase the angle to . How many complete turns around the circle have you made? How many repeats of the pattern has the graph made?
6. After how many degrees does the graph repeat itself?
7. What function is this a graph of?
8. Does the graph exist for angles smaller than and bigger than ? Are there any x-values that we cannot input into the sine function? Therefore, what is the domain of the sine function?
9. What is the range of the graph that has been produced?
10. What are the turning points between Are these maximum or minimum turning points or both? How far apart are the maximum turning points? How far apart are the minimum turning points?
11. What are the x-intercepts?
12. What are the y-intercepts?
13. Visit the interactive simulation at <https://www.desmos.com/calculator/wj129wvrjw>. Again, you will find a circle with a radius of one and a slider with which to change the size of angle .
14. Change the size of angle . What function is graphed now?
15. Is it possible for the graph to keep repeating outside the interval that has been shown?
16. What is the domain of function? How does this relate to the domain of the sine function?
17. What is the range of the function?
18. After how many degrees does the graph repeat itself?
19. What are the turning points of the graph? How far apart are they?
20. What are the x-intercepts?
21. What are the y-intercepts?
22. In both the sine and cosine functions above, the range is .
23. Look at the graphs of both functions again. What horizontal line is the “middle” of each graph?
24. What is the maximum distance away from this “middle” line that both graphs get?
25. How could we change the functions of and to have a range of and what would the maximum distance away from the “middle” line be then?
26. Make a rough sketch of these graphs for the interval .

#### Guided Reflection

1. The interactive simulation shows us the relationship between the size of the angle in circle and the angle that is being used as the input for a trigonometric function.
2. If we start at and turn one whole revolution or , then we get to . Another revolution will get us to . A further two revolutions will get us to . So, in total, it will take us 4 revolutions to get from to .
3. From to is one revolution around the circle. We can see that the graph completes one full pattern and starts another repeat of the same pattern.
4. From to requires two revolutions around the circle. By , the function has completed two patterns.
5. If we increase the angle to we will have made a total of four revolutions around the circle and the graph will have repeated its pattern four times.
6. The graph repeats its pattern after .
7. This is the sine function.
8. The graph will keep on repeating itself every forever for angles smaller than and larger than . There are no values of that we cannot input into the function. Therefore, the domain of the sine function is all real numbers or .
9. The range of the graph is all real numbers between and or or .
10. The graph has infinitely many turning points, both maximum and minimum. In the interval , the graph has maximum turning points at and . The graph has minimum turning points at and . All the maximum turning points are apart. All the minimum turning points are also apart.
11. The x-intercepts are and .
12. There is only one y-intercept at .
13. This interactive simulation graphs another trigonometric function.
14. This time, the graph is of the cosine function.
15. The graph will repeat its pattern for angle less than and greater than .
16. The domain of the cosine function is the same as the sine function. It is all real numbers or .
17. The range of the cosine function is the same as the sine function. It is all real numbers between and or or .
18. Just like the sine function, the graph of the cosine function repeats itself every .
19. The graph has infinitely many turning points, both maximum and minimum. In the interval , the graph has maximum turning points at and . The graph has minimum turning points at and . All the maximum turning points are apart. All the minimum turning points are also apart.
20. The x-intercepts are and .
21. There is only one y-intercept at .
22. Here are images of the sine and cosine functions for the interval .

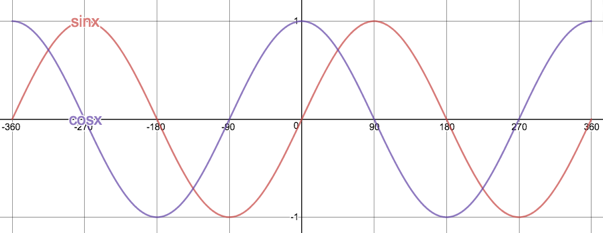
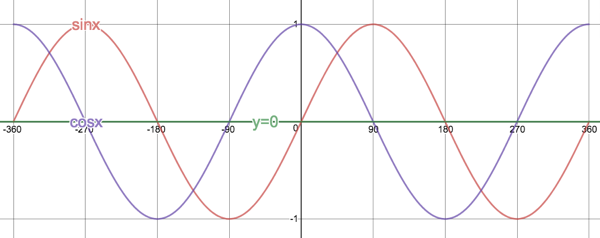
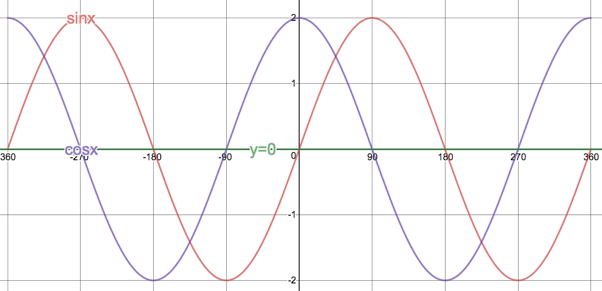


Image source: <https://www.desmos.com/calculator/lqbgox4miq>

1. For both functions, the horizontal line is the “middle” of the graph. Here is the same image but, this time, with this horizontal “middle’’ also shown. See how this line is exactly half way between the maximum and minimum turning points?

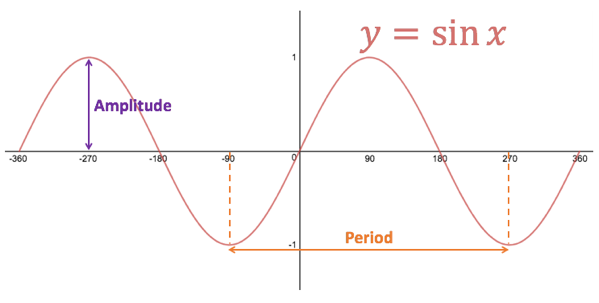


1. Both graphs get a maximum distance of one unit away from this middle line.
2. We know from earlier work on functions that we can vertically stretch (or squash) a graph by multiplying every output value by some constant. In other words, to stretch the function we could multiply every output value by two to get a new function . In the case of the sine and cosine functions, the new stretched functions will be and . Because every output or y-value will be multiplied by 2, the turning point will all have y-value of or . This means that the range will be and the maximum distance the graphs will get from their “middle” lines will be two units.
3. This is what the graphs of and look like.



We saw in the last activity that the functions and repeat themselves every . We call this their **period**. Both functions have a period of . The period of a sine or cosine function is the number of degrees it takes for the function to start repeating its pattern. You can measure the period of a sine or cosine function by measuring the difference in the x-values of any two sequential repeated points.

We also saw that both functions always exist within the range . In other words, they never get further than 1 unit away from the x-axis (the line ). We say that they each have an **amplitude** of 1. The amplitude is the maximum distance the function gets from its “middle”. In the case of and this “middle” is the x-axis.



### Activity 3: and

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will explore the effects of and on the functions and . |
| Stopwatch | Suggested Time You will need about 60 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. Draw a sketch of on a piece of paper for .
2. Mark in the turning points and show the amplitude and period of the function.
3. Now think about the function for ? What will its turning points be? What will its amplitude be? What will its period be?
4. Think about the function for ? What will its turning points be? What will its amplitude be? What will its period be?
5. Draw sketches of and on the same set of axes as , showing the amplitude and period of and as well.
6. Now visit the interactive simulation at <https://www.desmos.com/calculator/jj1awz5vfg> to check that your sketches are all correct.
7. What would the graph of look like? Check using the interactive simulation by changing the value of in .
8. Based on what you learnt in question 1), draw the following functions on the same set of axes - , and for .
9. Write down the domain and range of each function.
10. What is the amplitude and period of each function?
11. What are the turning points of each function?
12. Look at the following functions.

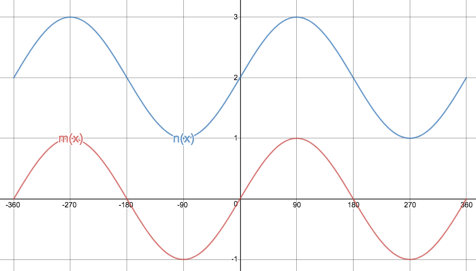


Image source: <https://www.desmos.com/calculator/3ekzjhnqrv>

1. What is the equation of ?
2. What is the amplitude of ?
3. What are the turning points and y-intercept of ?
4. What are the turning points and y-intercept of ?
5. Write down the equation for .
6. What is the amplitude of ?
7. Make neat sketches of the following functions on the same set of axes for . For each function, write down the domain, range, amplitude, period and list the turning points and y-intercept. Hint: It might help to make light sketches of the basic sine and cosine functions and then to transform these are required by each of the functions.

#### Guided Reflection

1. We needed to make a sketch of for the interval .
2. Here is the graph of with its turning points, period and amplitude shown.

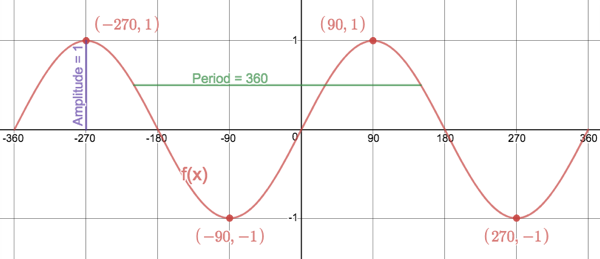


Image source: <https://www.desmos.com/calculator/hsomsg5wo0>

1. The function for is really the function . Therefore, its turning points will be:

Max: and

Min: and

The amplitude of the graph will thus be 2 but its period will be unchanged i.e. .

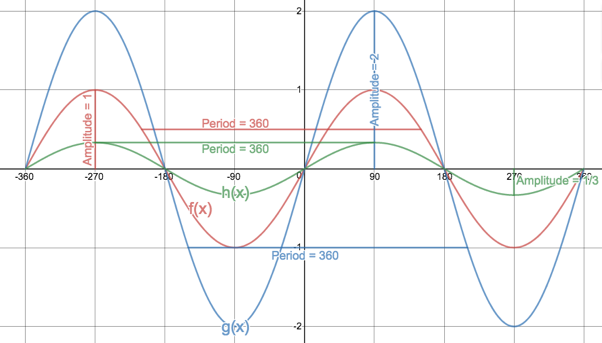
1. The function for ? Is really the function . Therefore, its turning points will be:

Max: and

Min: and

The amplitude of the graph will thus be but its period will be unchanged i.e. .

1. Here is a sketch of and .



1. Here are sketches of and .

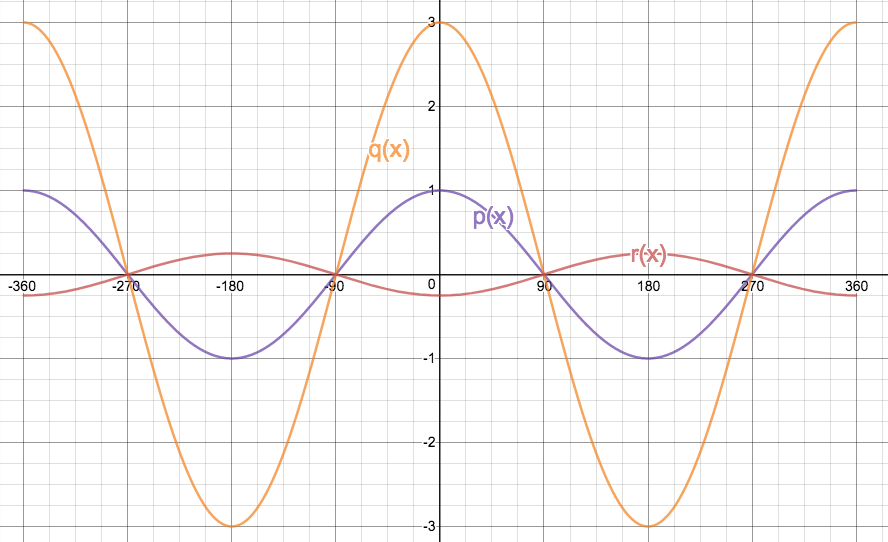


Image source: <https://www.desmos.com/calculator/l4yhruehjf>

1. : Domain Range:

: Domain Range:

: Domain Range:

1. : Amplitude: 1 Period:

: Amplitude: 3 Period:

: Amplitude: Period:

1. : and

: and

: and

1. We were given two functions, and .
2. The function is the basic sine function i.e. .
3. The amplitude of is 1.
4. The turning points of are and . The y-intercept is .
5. The turning points of are and . The y-intercept is .
6. The graph of has been shifted vertically up by 2 units. Therefore, we know that . In other words, .
7. The amplitude of is 1. The middle of the graph (the point half way between the min and max turning points) is the line . The maximum distance the graph gets from this “middle” is one unit.
8. Here are sketches of each of the functions.

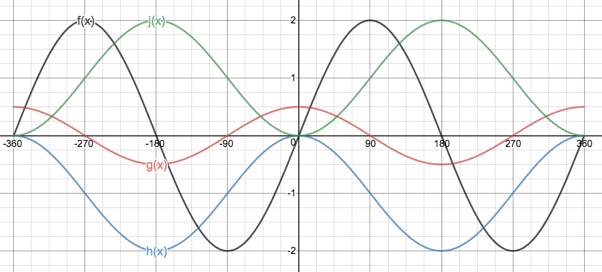


Image source: <https://www.desmos.com/calculator/ofvsuep3l2>

The function is the same basic shape as , except that it has an amplitude of 2. Therefore, the graph is stretch out vertically so that the turning points have y-coordinates of 2 and -2.

The function is the same basic shape as , except that the graph has been vertically squashed. Instead of the turning points having y-coordinates of 1 and -1, they now have y-coordinates of and .

The graph of is the graph of shifted down by 1 unit. Therefore, the y-coordinates of the turning points are all going to be one less than before i.e. the turning point of is now going to be and the turning point is now going to be .

The graph of has undergone three different transformations. Firstly, the graph will have an amplitude of 2 units. But, secondly, the graph ill also be upside down because the value of is negative i.e. what was a maximum turning point is not going to be a minimum turning point and visa verse. Thirdly, the graph has been shifted 1 unit up. So, all the y-coordinates of the transformed turning points are also going to be 1 unit greater.

For example, the turning point that was at is going to be transformed in the following ways:

because the absolute value of .

because .

because

domain, range, amplitude, period and list the turning points and y-intercept.

1. : Domain: Range:

Amplitude: 2 Period:

Turing points:

Y-intercept:

1. : Domain: Range:

Amplitude: Period:

Turing points:

Y-intercept:

1. : Domain: Range:

Amplitude: Period:

Turing points:

Y-intercept:

1. : Domain: Range:

Amplitude: Period:

Turing points:

Y-intercept:

The last activity showed us that the values of and in and affect the shape and position of the graphs in very much the same way as they do in other functions we have studied like , and .

The value of determines the vertical (up or down) shift of the graph and, therefore, its y-intercept. If we increase the value of , we shift the graph up. If we decrease the value of , we shift the graph down. Shifting the graph up and down affects the range of the functions.

Here is a summary of the effect of of the vertical position of the cosine function.

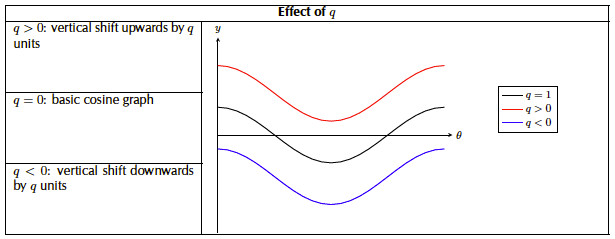


Image source: Everything Maths Gr10 (pg 194)

The value of determines the shape and orientation of the graph. In the case of the sine and cosine functions, this is expressed in terms of changes in the amplitude of the graphs. The bigger the absolute value of gets (i.e. ignoring the sign), the greater the amplitude. When , the graphs are flipped upside down.

Here is a summary of the effect of on the sine function.

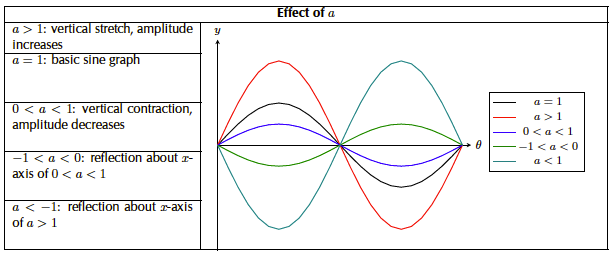


Image source: Everything Maths Gr10 (pg190)

### Activity 4: The Effect of

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will explore the effects of on the functions and . |
| Stopwatch | Suggested Time You will need about 45 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

1. Have a look at these graphs of and .

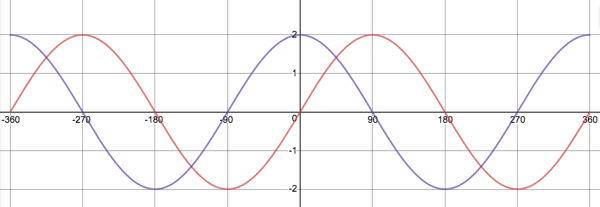


Image source: <https://www.desmos.com/calculator/kvrbn4xrbe>

1. Which graph is ? Why?
2. What are the values of and ?
3. What is the domain and range of each function?
4. What would we have to do to the sine function so that it produces the same graph as the cosine function?
5. Now visit the interactive simulation at <https://www.desmos.com/calculator/qbmsi5x12d> and write down two possible equations for a transformed sine function, , that has the same graph as the cosine function. Describe the transformations that would be necessary in each case.
6. Write down the domain, range, amplitude and period of these transformed sine functions. Are they different to the original sine function, ?
7. If the following is a graph of a cosine function of the form , what are the values of and . If you need to, draw this graph and the basic cosine function on the same set of axes.

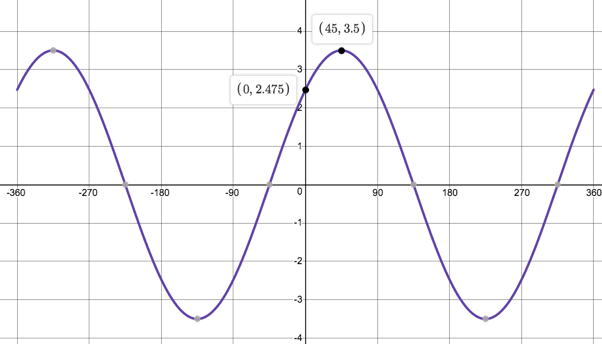
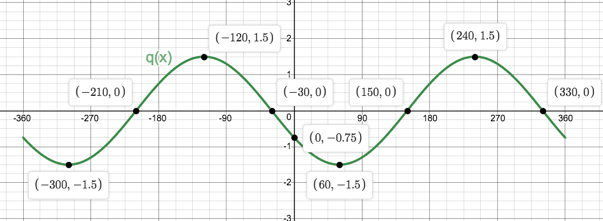


Image source: <https://www.desmos.com/calculator/ygszjpmnxr>

1. Have a look at the graph of the function .



1. What is the domain and range ?
2. What is the period of ?
3. What is the amplitude of ?
4. Write down three possible equations for the function giving reasons for each.
5. If you were told that and that , write down the equation of .

#### Guided Reflection

1. We were given and .
2. Based on the fact that the red graph is the one with a y-intercept of , it is the graph of .
3. Both graphs have an amplitude of 2. Therefore, the values of both and are 2.
4. The domain of both functions is . The range of both functions is .
5. We could do two things. Either we could shift the sine function *to the left* by or we could shift it *to the right* by .
6. – the original sine function has been shifted to the left.

– the original sine function has been shifted to the right.

1. :

Domain: Range:

Amplitude: 2

Period:

:

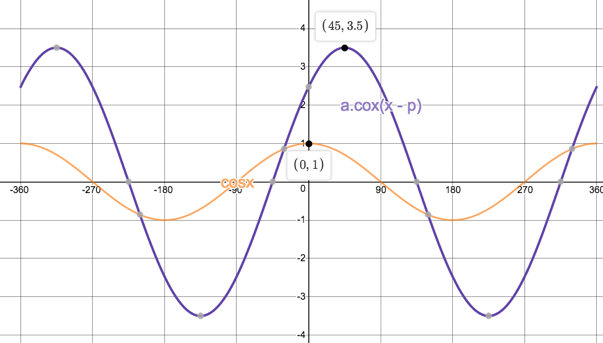
Domain: Range:

Amplitude: 2

Period:

In both cases, these are the same as the original function, .

1. Here are the graphs of and on the same set of axes.



We can see that the amplitude of is times higher than the amplitude of . Therefore, .

The TP of at has been shifted to i.e. it has been shifted *to the right*. Therefore, .

1. We have been given the function .
2. Domain: Range:
3. Period:
4. Amplitude: 1.5
5. – the amplitude of the graph is so . There is no vertical shift so . The basic cosine function normally cuts the y-axis and has a TP at . But this TP has been shifted to the left. Therefore .

– the amplitude of the graph is so . There is no vertical shift so . The basic cosine function normally cuts the y-axis and has a TP at . But this TP has been shifted to the right. Therefore .

– the amplitude of the graph is so . There is no vertical shift so . The basic sine function normally has a TP at . However, the TP is now at which means that the graph has been shifted to the right. Therefore, .

1. We are told that and that . Therefore, which means that the graph has been flipped upside down. What used to be maximum turning points, will now be minimum turning points and visa verse.

The basic sine function has a maximum TP at . The graph of has a minimum TP at . The difference in the y-coordinates is due to . The horizontal shift of to the left must be due to . Therefore, and .

As you probably expected, the effect of in and is to shift the graph horizontally (left and right). Once again, this is exactly the same as the effect of in other functions like , and .

* For : The graphs are shifted to the *left*.
* For : The graphs are shifted to the *right*.

However, we have also discovered that there are multiple sine and cosine functions that can generate the very same graphs. For example

all produce the same graph. This means that, for every input, each of these functions will produce the very same output. We will learn more about this in Topic 3.

### Activity 5: Periods

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will help you to understand the periods of the sine and cosine functions in more detail. |
| Stopwatch | Suggested Time You will need about 75 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

1. Here are three sine functions, , and for the domain with the coordinates of their maximum turning points.

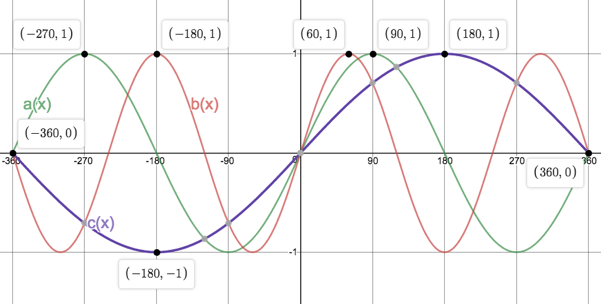


Image source: <https://www.desmos.com/calculator/8ptmdmayli>

1. Which function has the equation ?
2. What is the amplitude of each function?
3. What is the period of each function?
4. Write the periods of and in terms of the period of e.g. the period of some number the period of .
5. Now write the period of in terms of the periods of and e.g. e.g. the period of some number the period of .
6. If , what are the values of and if and ?
7. Write the periods of and in terms of the period of the basic sine function () and the values of or .
8. Complete the following table for

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

1. Sketch the graph of for
2. What is the period of ?
3. If is in the form express the period in terms of and .
4. For , sketch the functions , and on the same set of axes. Write down the domain, range, amplitude and period of each function.

#### Guided Reflection

1. We were given the graphs of , and .
2. . It is the only one that increases to a maximum TP at .
3. Each function has an amplitude of 1.
4. The period of is .

The period of is . The x-axis distance between, for example, the two successive maximum turning points and is .

The period of is . The x-axis distance between the two x-intercepts where the graph is increasing, and is

1. Period of / period of is . Therefore, the period of the period of .

Period of / period of is . Therefore, the period of the period of .

1. From our equations in d), we can say that

Period of the period of .

Period of the period of .

1. We know that , and all give the same output. We also know that

. Therefore so the value of in must be .

. Therefore so the value of in must be .

1. Period of or

Period of or

1. Here is the completed table.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

1. Here is a sketch of .

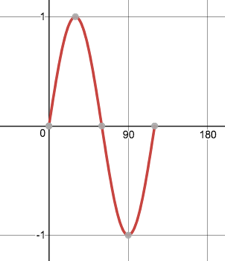
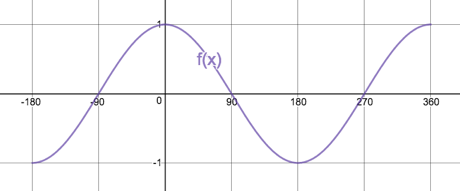


Image source: <https://www.desmos.com/calculator/s4lubpjiqa>

1. The period of is .
2. The period of .
3. We were asked to sketch , and on the same set of axes for the domain .

Whenever we need to sketch a sine or cosine function, it is best to start with the basic function and then transform it as required by the values of or .

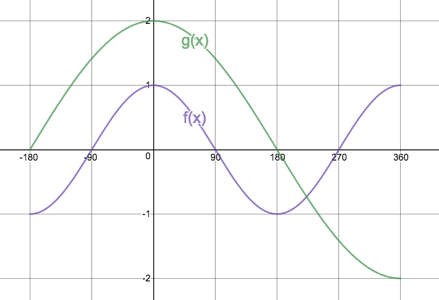
Here is a sketch of , the basic cosine function. Notice that we stop drawing at and .



Now . so, the graph will have an amplitude of 2. The values of and are zero, so there are no vertical or horizontal shifts.

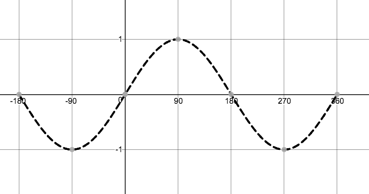
. This means that the period of will be . So, the turning point of at is now going to be twice as far from the y-axis at . The x-intercept of at is now going to be twice as far away from the y-axis at .

Here is a sketch of and .



is a sine function. It is a good idea to draw the basic sine function (even if just as a light or dotted line) to help you figure out what looks like.

Here is a sketch of the basic sine function for the interval.



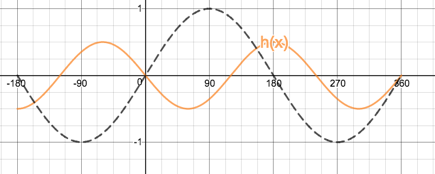
Now . so, the graph will have an amplitude of but will also be flipped over (maximum turning points will become minimum turning points).

and so there are no vertical or horizontal shifts.

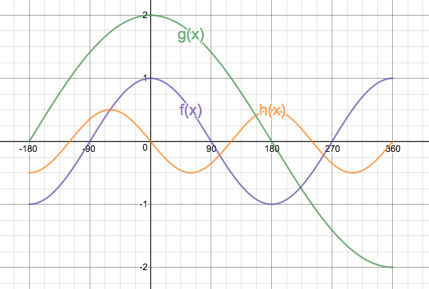
. This means that the period of will be . Therefore, the maximum turning point at will become a minimum turning point at . Remember has changed the amplitude and the orientation and has changed the period. A point that was at is now going to be at .

This also means that the x-intercepts at and will become , and or , and .

Here is a sketch of the basic sine function and .



Finally, here is a sketch of all three functions.



We were also asked to note the domain, range, amplitude and period of each function.

The domain of all three functions is the same as the specified interval i.e. domain is .

Range:

Amplitude:

: 1 : 2 :

Period:

: : :

So far, we have investigated the sine and cosine functions of the following forms:

We have seen that these two functions are very similar. In fact, we saw right at the start of this unit that . The fact that these two functions are so closely related is one of the reasons for how they have been named – sine and **CO**sine. Cosine is the co-function of sine.

But there is a third trigonometric function that behaves quite differently from sine and cosine. It is called tangent (tan for short). but the graph of looks nothing like the other two. Let’s take a closer look.

### Activity 6: The Tangent Function

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will introduce you to the graph of the tangent function. |
| Stopwatch | Suggested Time You will need about 35 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

1. We are going to try and make a sketch of .
2. Using a calculator, complete the following table. Does anything seem strange to you?

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |

1. Plot these points on a Cartesian Plane. Can you join them with a smooth curve.
2. The graph may seem to be a strange shape so calculate the following points and use them to help you draw the graph.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |

1. Use your calculator to try and work out what happens to the value of as you let get closer and closer to from the left and the right of . What happens when . For what other functions have you seen this kind of behaviour? What did we call this line?
2. Why is undefined? Think about the fact that .
3. For what other inputs do you see the same behaviour as you do for ?
4. Add any lines you need to complete your sketch of
5. What is the period of
6. What is the amplitude of ?
7. Is a function? Why?
8. What is the domain and range of ?

#### Guided Reflection

1. We need to make a sketch of .
2. Here is the completed table of values. The value of is undefined for and . The graph does not exist at these points.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |

1. Here are the points plotted. Joining them is hard because they don’t all seem to line on the same line

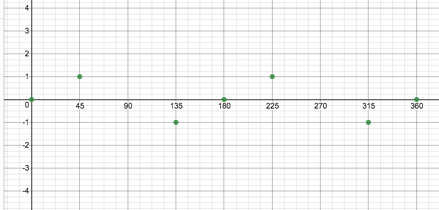


Image source: <https://www.desmos.com/calculator/bkged06usn>

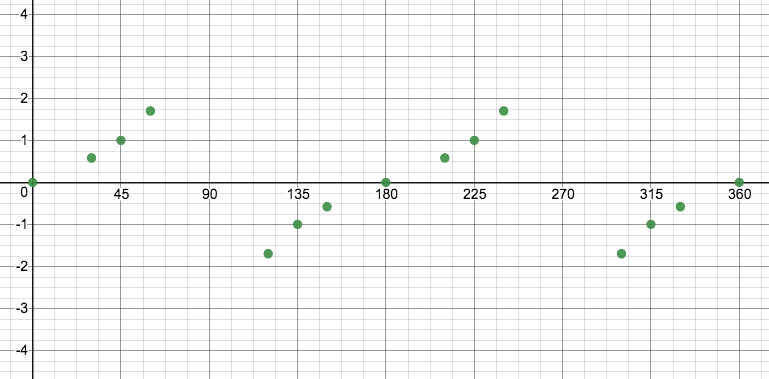
We could join them like this to make shark teeth.



1. Here is the second table of values.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |

If we add these new points, we get



But it is still not clear exactly how they need to be joined, especially the points either side of and .

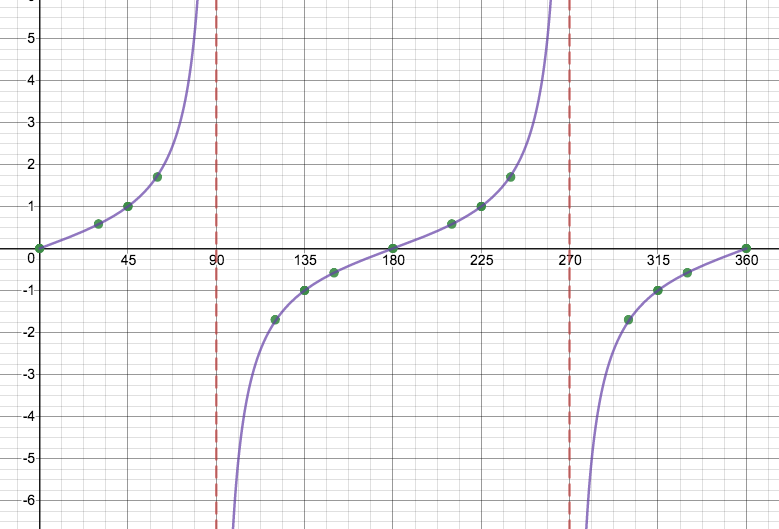
1. As we let get closer and closer to from the left, gets huge. For example, . When we let get closer and closer to from the right, gets hugely negative. .

We cannot let ever actually equal because then is undefined.

We saw similar behaviour with the hyperbolic and exponential functions. Both of these functions had asymptotes – values that the graph could get closer and closer to but was never allowed to reach. The hyperbolic function has a vertical and horizontal asymptote while the exponential function had a horizontal asymptote.

It looks like the tangent function has a vertical asymptote at .

1. If then and we know that we cannot divide by zero. That means that is undefined whenever .
2. The same asymptotic behaviour happens at .
3. If we draw in the asymptotes, we get a much clearer picture of the shape of the graph.



1. We can see that the graph repeats itself after . Therefore, the period of is .
2. Because the graph does not have a maximum or minimum value, it does not have an amplitude. The concept of amplitude does not apply to .
3. is a function. There is no input that ever produces more than one output. The vertical line test shows this.
4. The tangent function can take any real value for except the values where it has a vertical asymptote. From our graph, these asymptotes appear every . We can write the domain as . This means that can be any real number except , , and so on.

The range of the function is any real number i.e. .

### Activity 7: Transforming the Tangent Function

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will investigate the effect of the variables and on the function . |
| Stopwatch | Suggested Time You will need about 75 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook * An Internet connection |

#### Tasks

1. Visit the interactive simulation at <https://www.desmos.com/calculator/umnukoayzm>. Here you will find the graph of the basic tangent function in the form of and sliders to change the values of , , and . Also shown are the graph’s vertical asymptotes as well as the some points where the value of (in blue) and the x-intercepts (in purple).
2. If , what are the values of , , and ?. What is the period of the function now?
3. From your knowledge of the sine and cosine functions, which variable do we need to change to change the period of ? What is the value of this variable that changes the period of to ?
4. When we make the period of , how far apart are the asymptotes? How far apart are the x-intercepts.
5. What value of the variable would make the period of . Write down an expression that links the period of with this variable.
6. Make the period of again. From your knowledge of the sine and cosine graphs, what will be the effect of changing the value of in ?
7. What happens to the graph of if you make the value of ? Which points on the graph change? Which points do not change?
8. Change the value of to 0.3. What happens to the graph?
9. What will happen to the graph if ? Predict the coordinates of these special points and then change the value of in the interactive simulation to see if you are right.
10. Make again. From your knowledge of the sine and cosine graphs, what will be the effect of changing the value of in ?
11. What happens to the graph of if you make the value of ? Which points on the graph change and how?
12. Make again. From your knowledge of the sine and cosine graphs, what will be the effect of changing the value of in ?
13. What happens to the graph of if you make the value of ? Which points on the graph change?
14. What else changes when you change the value of in ? Write an expression that links the value of with this change.
15. If the points and are points on , what do the values of and need to be to make these points and respectively. Use the interactive simulation to check.
16. If the point is a point on , what do the values of and need to be to make this the point . Use the interactive simulation to check.
17. Sketch the graph of .
18. Sketch the graph of .
19. Sketch the graph of .

#### Guided Reflection

1. The interactive simulation at <https://www.desmos.com/calculator/umnukoayzm> allowed us to manipulate .
2. When , then , , and . The period of is .
3. We know that the value of in, for example, changes the period of the graph. Therefore, it is the value of in that will change the period of the tangent function. We know that if we make in , we will halve the period of the basic sine function from to because the period is given by (remember, the sign of does not matter). So, if we want to double the period of the basic tangent function from to , we need to make .
4. The asymptotes and x-intercepts are now apart. This is because the graph repeats itself only every .
5. To make the period of equal to , the value of because . Therefore, in general we can say that the period of the tangent function is given by .
6. Changing the value of will stretch (or squash) the graph vertically, making it longer and thinner or shorter and fatter.
7. When , all the points that used to have y-coordinates of now have y-coordinates of , The graph has been stretched out. The x-intercepts do not change at all.
8. When we make we squash the graph vertically. All the points that used to have y-coordinates of now have y-coordinates of .The x-intercepts do not change at all.
9. If we make , two things will happen. The y-coordinate of these special points will change from in the basic tangent function to but they will all, also change sign. In other words, the point , for example, will become the point . The point will become .
10. We know that changing the value of shifts the graph vertically up or down.
11. If we make , we will shift the whole graph 2 units *down*. All the y-coordinates will be 2 units less. This includes the special points as well as the x-intercepts. Obviously, these points, being 2 units further down then before, are no longer x-intercepts.
12. Changing the value of shifts the graph horizontally left or right.
13. If we make , we shift the whole graph to the *right*. The x-coordinate of every point on the graph moves to the right. For example, the point becomes the point .
14. The vertical asymptotes also shift to the right. The asymptote at becomes an asymptote at . In general, this asymptote becomes the line . The other asymptotes, like shift in the same way i.e. .
15. We know that . Therefore, and . So, the period is and there is no horizontal shift happening. We know that, normally, the points and are the same distance from the x-axis but our new points are not. This tells us that there is a vertical shift. We know that for , the vertical distance between these points is 2. However, now the vertical distance is 3. This means that the value of and that .

Another way to think about this is that each of these points is 1 unit () away from the “middle” of the graph (similar to the amplitude of sine and cosine). But the new points are or units from the “middle” of the graph.

However, the vertical shift has moved this “middle” up. The middle line between the y-coordinates and is the line . Therefore, the graph has been shifted up by 1 unit so . The equation of the function is thus .

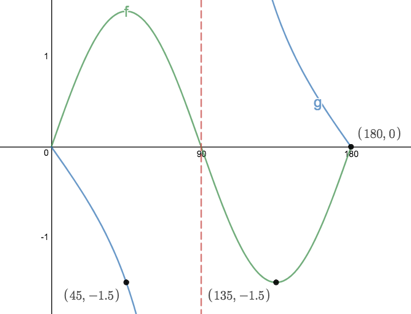
1. We know we are working with the function where and . This means that the period of the graph is . To move a point from to requires a vertical shift *down* of three units and a horizontal shift to the *left* of . Therefore, and and the equation is

### Activity 8: Trigonometric Functions

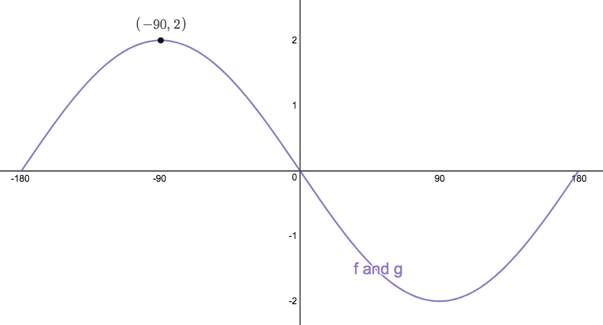
|  |  |
| --- | --- |
| Bullseye | Purpose This activity will consolidate your knowledge of the trigonometric functions. |
| Stopwatch | Suggested Time You will need about 60 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

1. Sketch the graph of for .
2. Sketch the graph of for .
3. Sketch the graph of for .
4. and What are the equations of these functions.



1. This is the graph of and What are the values of , and ?



1. Given the functions and .
2. Sketch the functions on the same set of axes for the interval .
3. What is the period of ?
4. What is the amplitude of ?
5. Use your sketch to determine how many solutions there are to the equations in the interval and give one of these solutions.
6. Indicate on your sketch where the solution to is found.
7. Show on your graph where ?
8. For what values is ?

#### Guided Reflection

1. We were asked to sketch the graph of . Like when sketching sine and cosine graphs, it is best to start out with a light sketch of eth basic tangent function and then transform this graph needed. The x-intercepts, the vertical asymptotes and the points and are all very useful markers.

In our function, . That means that the period of the graph is going to be or three times greater. So instead of the asymptotes being apart they are now going to be apart. The vertical asymptote that used to be at is now going to be at and the vertical asymptote that used to be at is now going to be at .

Also, the value of . This means that the graph is going to be squash vertically. Therefore, the point that used to be is now going to have a y-coordinate of and because the period is now three times greater, it will have an x-coordinate of i.e. it will be the point ). The point that was at is now going to be at .

Here is a sketch of the graph. The basic tangent graph is shown as a dotted line.

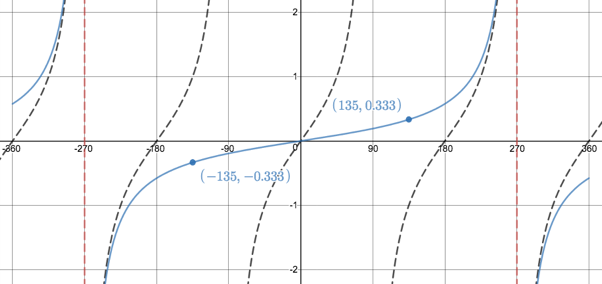


Image source: <https://www.desmos.com/calculator/zftcpmc1uc>

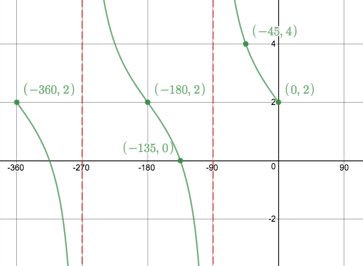
1. We need to sketch for . As normal, start with a light sketch of the basic tangent function, paying particular attention to the key points in the required interval.

In , the value of . This means that the graph is going to be stretched vertically and is going to be reflected about the y-axis. The value of so the period of the graph is as normal and the vertical asymptotes are where they normally are - and .

The transformations because of the value of mean that the point becomes the point and the point becomes the point .

But which means that the whole graph has also been shifted 2 units up. Thus, every point, including the x-intercepts is moved 2 units up. So, the x-intercept becomes the point , the x-intercept becomes the point and the x-intercept becomes the point The transformed points above are also shifted 2 units up and become and .

Here is a sketch of the graph.



1. We need to sketch for . As normal, start with a light sketch of the basic tangent function, paying particular attention to the key points in the required interval.

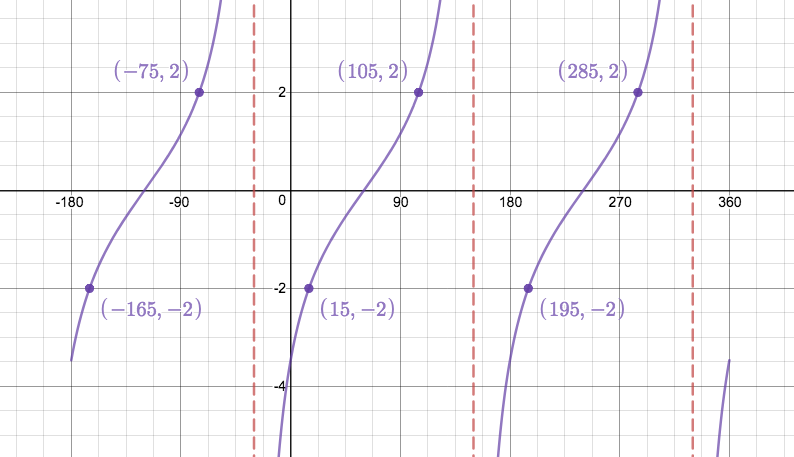
In , the value of . This means that the graph is going to be stretched vertically and points that used to have y-coordinates of will now have y-coordinates of .

The value of so the period of the graph is as normal and the vertical asymptotes are going to be apart as normal.

However, the horizontal shift to the right because is going to move the vertical asymptotes from to , from to and from to .

The horizontal shift will also move every other point on the graph to the right. Therefore, the x-intercepts will move to , will move to , will move to .

Here is a sketch.



1. We are told that and From the graph, we can see that the period of is . Therefore, we know that the value of .

The minimum turning point of has a y-coordinate of rather than its normal value of 1. So, we know that the graph has been stretched vertically and the value of . Therefore, .

The graph of decreases from zero to the vertical asymptote at instead of increasing as normal. Therefore, we know that the value of . The y-coordinate of the point at is . We already know that so we can conclude that . Therefore,

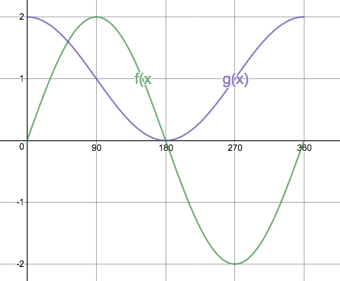
1. We are told that the give sketch is of and Let’s start by working with the sine function.

The period of the graph is so we know that . The sine function normally increases between and but this one is decreasing. This means that . There is a turning point at . Thus, the amplitude of the graph is and .

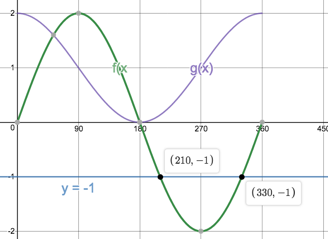
Working with the cosine function. We know that cosine normally has a maximum turning point and y-intercept at . However, the turning point has been shifted to the left by . This means that . The fact that the amplitude of the graph is 2 units means that .

This result means that .

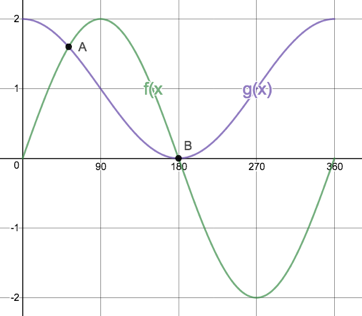
1. We were given and
2. Here are sketches of these functions for the interval .



1. The period of is .
2. The amplitude of is 1.
3. We can write as . In other words, this is . Because the graphs intersect each other twice in this interval, we know that there are two solutions. One of these solutions is the point .
4. The solution to is where the graph of intercepts with the line as shown on the graph. Another way of saying this is that the solutions to the equation exist wherever .



1. means that, for a particular value of , the output from the function is greater than the output from the function . In other words, the graph of is higher than the graph of . We can see that this is always the case between points A and B but not including these points because this is where .



1. For the outputs of and must be opposite in sign i.e. if is positive, then must be negative and if is negative, then must be positive. The only time this happens in our interval is between where and . or because at these points one of the functions is zero and so their product will also be zero and we were asked for “less than” zero and not “less than or equal to” zero.

### Activity 9: Interpreting Functions!

|  |  |
| --- | --- |
| Bullseye | Purpose This activity will consolidate your knowledge of all the functions we have explored and give you practice in interpreting what the functions are telling us. |
| Stopwatch | Suggested Time You will need about 60 minutes. |
| Pencil | What You Will Need  * A pen or pencil * Some blank paper or a notebook |

#### Tasks

Exercises from Everything Maths Gr11 (pg237)