NASCA Mathematics Materials Draft 1

## Topic 3: Measurement, Euclidean Geometry, Analytical Geometry and Trigonometry

## Analytical Geometry

## Unit 1: The distance between two points

## Learning Outcomes

By the end of this unit, learners should be able to

## Use the Cartesian co-ordinate system to derive and apply a formula to calculate the distance between any two points (*x*1; *y*1) and (*x*2; *y*2)

## Activity 1: Horizontal and vertical distances

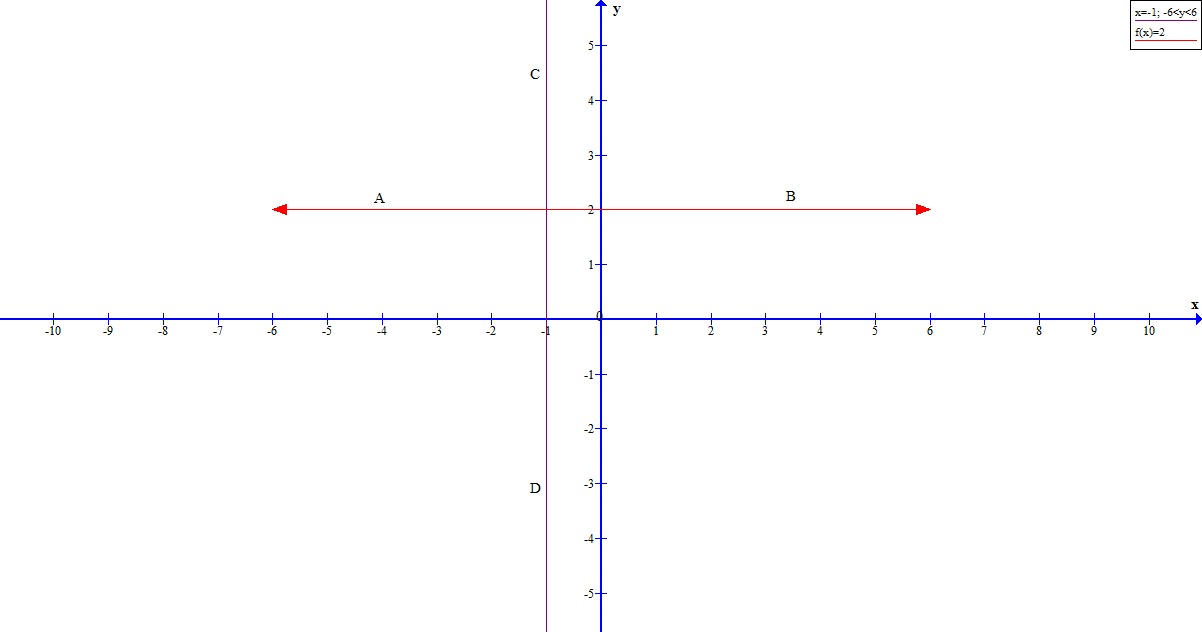
**Purpose**:

* Develop conceptual understanding of vertical and horizontal distances
* Calculate horizontal distance between 2 points
* Calculate the vertical distance between 2 points

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 20 minutes

**Introduction**



Line AB is a horizontal line in the Cartesian plane as it is parallel to the -axis.

Line CD is a vertical line Cartesian plane as it is parallel to the -axis

Any horizontal line in the Cartesian plane has the general solution

, where *c* is the y-intercept.

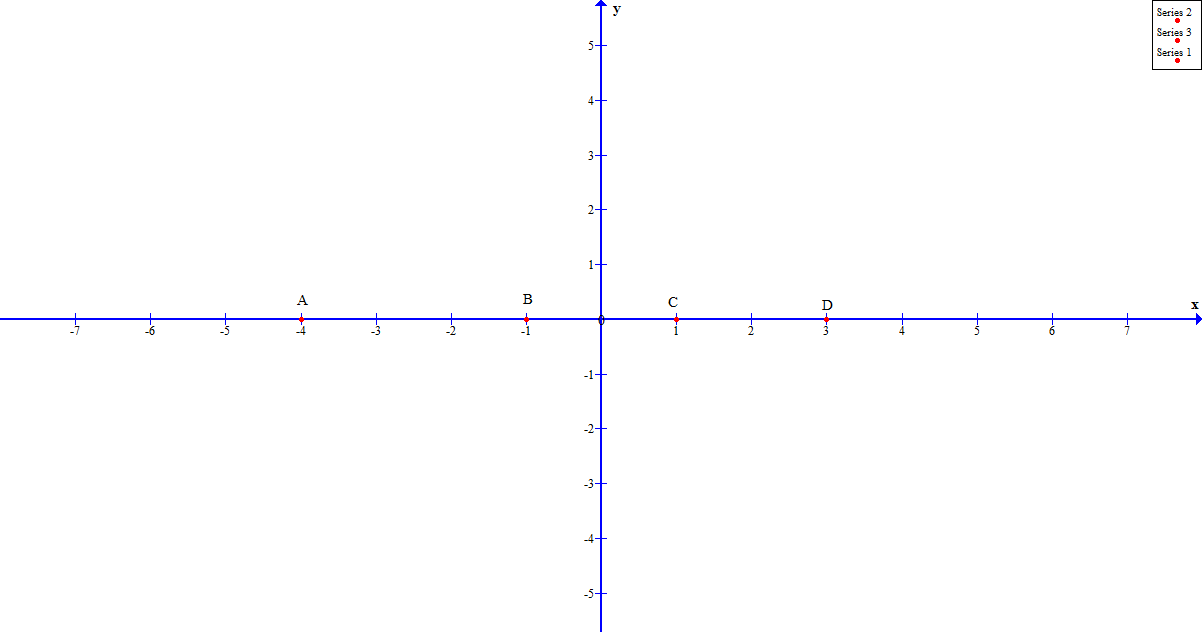
The equation of the horizontal line AB is .

Any vertical line in the Cartesian plane has the general solution c, where is the -intercept.

The equation of the vertical line CD is .

**Example 1**

, , ) and D) are four points on the -axis of a Cartesian co-ordinate system.



The distance between two points is zero or a positive quantity.

1. Determine the distance from C to D.
2. Determine the distance from B to C.
3. Determine the distance from A to B.

**Solution**

1. The horizontal distance CD = ( )

If and are two points on a horizontal line, then their ordinate values are equal, i.e.

=

=

1. The horizontal distance BC = ( )

The distance AB between the points and on a horizontal line is defined by their abscissa values as follows:

AB = if

AB = if

=

=

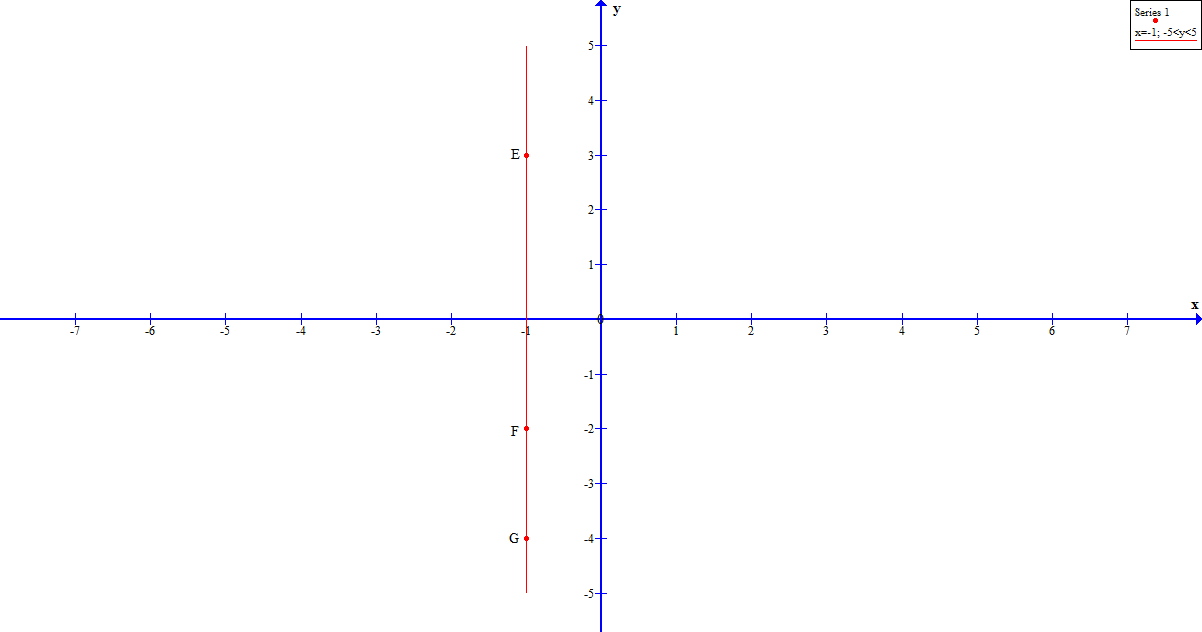
1. The horizontal distance AB = ( )

=

=

**Example 2**

, , ) are three points on a vertical line in the -axis in a Cartesian plane.



1. Determine the distance from E to F.
2. Calculate the distance from F to G.
3. Calculate the distance from G to E.

**Solution**

If and are two points on a vertical line, then their abscissa values are equal, i.e.

1. The vertical distance EF = ()

=

=

The distance FG between the points and on a vertical line is defined by their ordinate values as follows:

FG= if

GE = if

1. The vertical distance FG = ()

= -2-(-4) = 2

1. The vertical distance GE = ()

= 3-(-4) = 7

**TASK 1**

1. 

Calculate the lengths of AB and CD

#### Guided reflection on Task 1

|  |
| --- |
| 1. Why will you say that AB is a horizontal line?   The ordinate values of points A and B are equal, i.e.   1. Do you agree that the distance between A and B is equal to .   Yes   1. Why will you say that CD is a vertical line?   The abscissa values of points C and D are equal, i.e.   1. How can you find he distance between C and D?   By finding the value of |

**Answers to Task 1**

## Activity 2: Using Pythagoras theorem to derive the distance formula

**Purpose**: To derive the distance formula through using the theorem of Pythagoras

**Resources**: A pen or pencil, eraser, calculator and paper

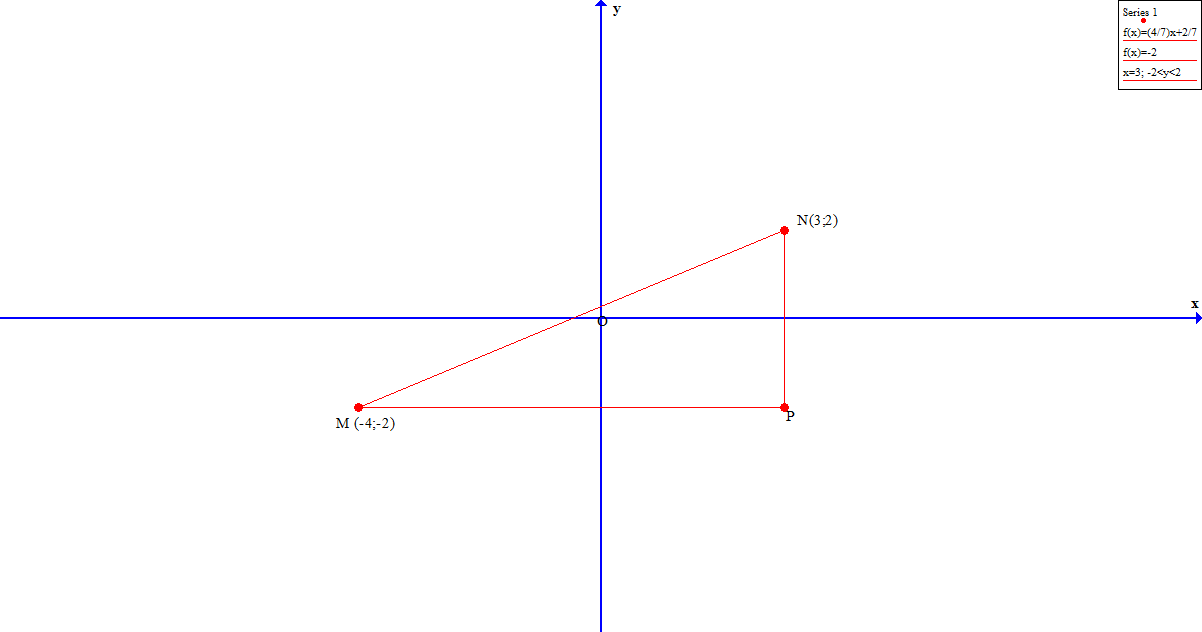
**Suggested time:** 30 minutes

**Introduction**

In this activity we work towards a formula to calculate the distance between any 2 points on the Cartesian coordinate system.

**Example 1:**

In the diagram below, M (-4; -2) and N (3; 2) are the endpoints of the line segment MN. A horizontal line through M and a vertical line through N are drawn to form , having = 900.



In space, the shortest

distance between two points

is via the straight line between the points. Hence, in mathematics, when we

talk about distance, we

actually refer this shortest

distance.

* 1. Use the diagram to complete the following statements:

1. P has the same -coordinate as the point …
2. P has the same -coordinate as the point …
3. The coordinates of P are (…;…)
   1. Determine the distance from M to P.
   2. Determine the distance from N to P
   3. Use the theorem of Pythagoras to calculate the distance between points M and N

.

**Solutions:**

* 1. (a) N

1. M
2. (3;-2)

1.2 Horizontal distance MP = ( )

=

= units

1.3 Vertical distance NP = ()

=

= units

1.4 MN2 = MP2 + NP2  Pythagoras

= 72 + 42

= 49 + 16

= 65

MN =  = 8,06 units

Looking back, we see that

=

=

=

= 

= 8,06 units

**Example 2:**

In the diagram below, A(*x*1; *y*1)and B(*x*2; *y*2) are the endpoints of line segment of length *d* units on the Cartesian plane.



* 1. Write down the coordinates of point C.

2.2 Determine the length of BC in terms of *y*2 and *y*1.

2.3 Determine the length of AC in terms of *x*2 and *x*1.

2.4 Copy and complete:

AB2 = … + …. (Pythagoras)

*d*2 = (*x*2 - *x*1)2 + …

The distance *d* between any two points (*x*1; *y*1) and (*x*2; *y*2)on the Cartesian plane is given by the distance formula,

**Solution**

* 1. C (

2.2 length of BC =

2.3 length of AC =

2.4 Copy and complete:

AB2 = AC2 + BC2 (Pythagoras)

*d*2 = (*x*2 - *x*1)2 +

**Activity 3: Applying the distance formula**

**Purpose**:

* To apply the distance formula to calculate the distance between any two points

on the Cartesian coordinate system.

* Use the distance formula to prove that a given figure is a specific type of triangle or quadrilateral.

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 45 minutes

**Example 1**

**Key words**

**Distance**: In space, the shortest distance between two points is via the straight line between the points. Hence, in mathematics, when we talk about distance, we actually refer this shortest distance.

Calculate the distance between A (-6;-4) and B (2; 6), correct to 2 decimal places.

**Solution:**

Let A be the point(*x*1; *y*1) and B the point (*x*2; *y*2).

AB =

=

=

=

=

=

=

= 12,81 units

**Example 2**

The points A (-2;1), B (4;4) and C (1;-2) are the vertices of a triangle. Determine whether ∆ABC is a scalene, isosceles, or equilateral triangle.

**Scalene triangle:**

A triangle whose sides are

all of different lengths is

called scalene. Its angles

are all different sizes.



**Isosceles triangle**:

An isosceles triangle is

a triangle with two sides

of equal length. It also has

two equal angles, which are opposite the equal side

**Solution**

***AC***  =

=

=

=

= units

**Equilateral triangle:**

An equilateral triangle is

a triangle whose sides are

all of equal length. Its angles

are all 60o.

***AB***  =

=

=

=

= units

***BC***  =

=

=

=

= units

∆ABC is isosceles, since two of its sides are equal in length, i.e. AB = BC.

**Example 3**

ABCD is a quadrilateral with vertices A(-3;4), B(2;1), C(-9;-6) and D(-4;-9).



* 1. Show that ABCD is a parallelogram.
  2. Is ABCD a rectangle? Justify your answer.

**Solutions**

1. AB =

=

=

DC =

=

=

**So AB = DC**

DA =

=

=

CB =

=

=

**So DA = CB**

ABCD is a parallelogram (both pairs of opposite sides of quad ABCD are equal)

**b)** DB =

=

=

In :

DA = (Proved in Question 1.1.)

AB = (Proved in Question 1.1.)

DB

(Converse of Theorem of Pythagoras)

ABCD is a rectangle because it is a parallelogram with a right angle.

**Task 1**

1. Use the distance formula to find the distance between the given points, correct to two decimal places wherever necessary:
2. A(0;0) and B(4;3)
3. P(-1;2) and Q(11;7)
4. T(4;-5) and U (7;9)
5. K(-6;2) and L( 6;-3)
6. D(9;0) and E(-5;-3)
7. The distance between K(-1;2) and L(t;4) is units. Find the value(s) of t.
8. D(-5;-3), E(-2;1) and F(-1;-6) are the vertices of ∆DEF.
9. Show that ∆DEF is isosceles.
10. Calculate the perimeter of the triangle, correct to two decimal places.
11. Determine whether the triangle whose vertices are given in each case is scalene, equilateral or isosceles:
12. A(-2;5); B(3;3) and C(1;-2)
13. G(0;5); H (5;0) and I (-5;0)
14. W(-1; ), B(2; - ) and A(5; )
15. TURF is a quadrilateral with vertices T(2;3), U(4;-1); R(0:3) and F(-2;-1).



1. Use the distance formula to show that TURF is a rhombus.
2. Now prove that TURF is a square

#### Guided reflection on Task 1

|  |
| --- |
| 1. Let A be the point(*x*1; *y*1) and B the point (*x*2; *y*2). Write down a formula to calculate the distance between A and B.   AB =   1. What do you know about the sides of an isosceles triangle?   Two sides are always equal   1. How will you show that a triangle is scalene if you are given the coordinates of the vertices of the triangle?   I will use the distance formula to calculate the lengths of each side of the triangle. Then if all sides are unequal, I will conclude that the triangle is scalene.   1. How will show that a quadrilateral is a rhombus if you are given the coordinates of the vertices of the quadrilateral?   The easiest will be to calculate the lengths of each of the quadrilateral, and if all sided are equal then you can conclude that it is a rhombus. |

#### Answers for Task 1

1. a. Let A be the point(*x*1; *y*1) and B the point (*x*2; *y*2).

AB =

=

=

=

=

= units

1. Let P be the point(*x*1; *y*1) and Q be the point (*x*2; *y*2).

PQ=

=

=

=

= units

= 13 units

1. Let T( 4;-5) be the point(*x*1; *y*1) and U (7;9) be the point (*x*2; *y*2).

*TU*=

=

=

=

= units

1. Let K(-6;2) be the point(*x*1; *y*1) and L (6;-3) be the point (*x*2; *y*2).

*KL*=

=

=

=

=

= 13 units

1. Let D (9;0) be the point(*x*1; *y*1) and E(-5;-3) be the point (*x*2; *y*2).

DE=

=

=

=

= units

1. Let K(-1;2) be the point(*x*1; *y*1) and L(t;4) be the point (*x*2; *y*2).

KT=

=

=

=

= units

But KT = units

=

1. (a) Let D(-5;-3)be the point(*x*1; *y*1) and E(-2;1)be the point (*x*2; *y*2).

DE=

=

=

=

=

= 5 units

Let D(-5;-3)be the point(*x*1; *y*1) and F(-1;-6)be the point (*x*2; *y*2).

DF=

=

=

=

=

= 5 units

DE = DF

∆DEF is isosceles, since two of its sides are equal in length.

1. From (a) above we DE = DF = 5 units. So let us calculate the length of EF

Let E(-2;1)be the point(*x*1; *y*1) and F(-1;-6)be the point (*x*2; *y*2).

EF=

=

=

=

=

= units.

Perimeter of ∆DEF = DE+EF +DF

= 5 + 5 +7,07

= 17,07 units.

1. (a) Isosceles, equal sides = 29 units.
2. Let G(0;5)be the point(*x*1; *y*1) and H(5;0)be the point (*x*2; *y*2).

GH=

=

=

=

=

= 5 units

Let G(0;5)be the point(*x*1; *y*1) and I(-5;0) be the point (*x*2; *y*2).

GI=

=

=

=

=

= 5 units

GH = GI

∆GHI is isosceles, since two of its sides are equal in length.

1. Equilateral, equal sides = 4 units
2. (a) TU =

=

=

UR =

=

=

FR =

=

=

**=**

FT =

=

=

**=**

**So TU = UR= FR = FT**

TURF is a rhombus (The sides of quadrilateral TURF are equal)

1. TR =

=

=

In :

TU = (Proved in Question 1.1.)

UR = (Proved in Question 1.1.)

TR

(Converse of Theorem of Pythagoras)

TURF is a square (TURF is a rhombus having a right angle)

**Summary Assessment**

1. Use the distance formula to find the distance between the given points, correct to two decimal places wherever necessary:

1. M( 2;3) and N (7;9)
2. E(-4,5;4,5) and F(-5,5;5,5)
3. The diagram shows the points E(0;2), F(4;5), G(4;0) and H(0;-3).



* 1. Calculate the length of HG.
  2. Is isosceles? Justify your answer.
  3. Prove that EFGH is a rhombus.

1. The coordinates of the vertices of quadrilateral ABCD are A(3;3), B(1;1), C(6;-4) and D(8;-2).
   1. Show that ABCD is a parallelogram.
   2. Hence, prove that ABCD is a rectangle.

#### Guided reflection on Summary Assessment

|  |
| --- |
| 1. Use the answers given for the summary assessment to check if your answers were correct. 2. Did you realize why you got some your answers incorrect? 3. Have you been able to correct errors you made? |

**Answers for summary assessment**

1. (a) Let M( 2;3) be the point(*x*1; *y*1) and N (7;9) be the point (*x*2; *y*2).

MN =

=

=

=

= units

* + 1. Let E(-4,5; 4,5) be the point(*x*1; *y*1) and F(-5,5; 5,5) be the point (*x*2; *y*2).

EF =

=

=

= units

1. (a) Let H(0;-3)be the point(*x*1; *y*1) and G(4;0) be the point (*x*2; *y*2).

HG=

=

=

=

=

= 5 units

1. Let H(0;-3) be the point(*x*1; *y*1) and E(0;2) be the point (*x*2; *y*2).

Note: Points H(0;-3) and E(0;2)are two points on the y-axis, which is a vertical line.

Hence, the vertical distance HE = , wher

= 2-(-3)

= 5 units

HE=

=

=

=

= 5 units

But HG = 5 units …. (Calculated in 4.1\_

HE = HG

∆HEG is isosceles, since two of its sides are equal in length

.(c) Prove that EFGH is a rhombus.

To prove the EFGH is a rhombus, we need to show that HG = HE = EF= FG.

We have already shown in 4.2 that HE= HG = 5 units.

Now let us calculate EF and HG to see if their lengths are also 5 units each.

Let F(4;5) be the point(*x*1; *y*1) and G(4;0) be the point (*x*2; *y*2).

Points F(4;5) and G(4;0) are two points on a vertical line.

Hence, the vertical distance FG = , where

= 5-(0)

= 5 units

Let E(0;2) be the point(*x*1; *y*1) and F(4;5) be the point (*x*2; *y*2).

EF=

=

=

=

=

= 5 units

HG = HE = EF= FG= 5 units

ABCD rhombus ( four equal sides)

.3. First draw a sketch to represent the given information.



1. Now, to show that ABCD is a parallelogram, we need to show that both pairs of opposite sides are equal. This means that we must show that AB = DC and AD = BC. One can use the distance formula to calculate the lengths of AB and DC as well as the lengths of CD and A respectively.

We first show that AB =CD:

Let A(3;3)) be the point(*x*1; *y*1) and B(1;1) be the point (*x*2; *y*2).

AB =

=

=

=

= units

Let D(8;-2) be the point(*x*1; *y*1) and C(6;-4) be the point (*x*2; *y*2).

DC=

=

=

=

= units

but AB = units …. (calculated above)

Secondly we show that AD =BC:

Let A(3;3) be the point(*x*1; *y*1) and D(8;-2) be the point (*x*2; *y*2).

AD =

=

=

=

= units

Let B(1;1) be the point(*x*1; *y*1) and C(6;-4) be the point (*x*2; *y*2).

BC=

=

=

=

= units

but AD= units …. (calculated above)

Now, in quadrilateral ABCD,

… (proved above)

and AD = DC … ( proved above)

is a parallelogram (both pairs of opposite sides are equal)

1. To show that that ABCD is a rectangle, it is necessary and sufficient to show that it is a parallelogram with a right angle. We have already shown in 5.1 that ABCD is a parallelogram.

All that remains for us to now show that parallelogram ABCD has a right angle. To do this we can consider , and use the converse of the Theorem of Pythagoras to show that . We proceed as follows:

We already have the length of AB= units and length of AB= units.

So,

Let us now calculate the length of BD as follows:

Let B(1;1) be the point(*x*1; *y*1) and D(8;-2) be the point (*x*2; *y*2).

BD=

=

=

=

= units

= 58 units

Now in :

(calculated above)

= 58 units (calculated above)

(converse of the Theorem of Pythagoras)

ABCD is a rectangle (A parallelogram with a right angle)

## Unit 2: The gradient of a line segment

## Learning Outcomes

By the end of this unit, learners should be able to

## Use the Cartesian co-ordinate system to derive and apply a formula to calculate the distance between any two points (*x*1; *y*1) and (*x*2; *y*2) ;

## Use the Cartesian co-ordinate system to derive and apply a formula to calculate the gradient of the line joining any two points (*x*1; *y*1) and (*x*2; *y*2) ;

## Use the Cartesian co-ordinate system to derive and apply formula to calculate the gradient of a line that is parallel to another line;

## Use the Cartesian co-ordinate system to derive and apply formula to calculate the gradient of a line that is perpendicular to another line;

**Activity 1: Understanding the concept of gradient**

**Purpose**:

* To develop understanding of concept of gradient using the Cartesian co-ordinate system.

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 15 minutes

**Introduction**

In the given picture, the mountain range

[](http://www.sa-venues.com/gallery/table-mountain-31.jpg)

is steeper at certain places than others.

The steepest parts of the mountain range

have the greatest gradient. In mathematics,

gradient is the measure of the steepness of

a line segment (or line)

In Figure 2.1, line segment AB join points A(-3;2)



and B(2;4). Line segment AB determines a special

right angle having a horizontal and a vertical leg.

The length of the vertical leg BC, called the

rise, is 6 units. The length of the horizontal leg AC,

called the run, is 5 units. The gradient of line

AB(or line segment AB), is the ratio of the lengths of Figure 2.1

these two legs, or the ratio of rise to run.

Hence, we have gradient of line segment .

In mathematics the symbol *m* is used to represent the

word gradient. Therefore, using symbolic notation,

we have .



As shown in Figure 2.2, the rise and run

can be calculated as follows:

rise = 4-(-2) = 6 units , and

run = 2-(-3) = 5 units.

So, basically one can determine the gradient Figure 2.2

of line segment AB as follows:

In mathematics, the rise is equal to the change in the *y*-values on the

vertical leg (called the vertical difference or the change in *y*), and the run is equal to the corresponding change in the *x*-values on the horizontal leg

(called the corresponding horizontal difference or the corresponding

change in *x*). So, therefore we can also calculate the gradient of line segment AB as follows:

In the above example, the line segment rises from left to right and has a positive gradient.

**Activity 2: The gradient formula**

**Purpose**:

* To use the Cartesian co-ordinate system to derive a formula to calculate the gradient of  
  the line joining any two points (*x*1; *y*1) and (*x*2; *y*2)
* Apply gradient formula to calculate the gradient of the line joining any two points (*x*1; *y*1) and (*x*2; *y*2)

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 45 minutes

In the previous example, we have established

that the gradient of a line segment

=



In the figure, line PQ passes through any

two points, *P*(*x*1; *y*1) and *Q*(*x*2; *y*2). By inspection,

the coordinates of point R are (*x*2; *y*1).

We calculate the gradient of the line segment

joining the points *P*(*x*1; *y*1) and *Q*(*x*2; *y*2) by

using the formula:

= …… [One should always keep the order in which you subtract the

coordinates of the points the same in both the numerator and denominator]

Note that =

=

Therefore, we can equivalently calculate the gradient of a line segment by using the following formula as well:

**Example 1:**

Find the gradient of the line segment joining the two points

T (-2; -3) and P(4;2).



**Solution:**

Let *T* be the point (*x*1; *y*1) and *P* be the point(*x*2; *y*2).

Then, we get: *x*1= -4 and *y*1= -2 as well as *x*2= 2 and *y*2 = -3.

We calculate the gradient of line TP, by using the formula

= , and making the appropriate substitutions:

=

=

=

= -

Do you think you will get a different answer for the gradient of *TP* if you let *T* be the point (*x*2; *y*2) and *P* be the point(*x*1; *y*1). Check your response by copying and completing the following solution in your workbook:

=

=

= ……..

= ……...

In the above example, the line segment rises from right to left and has a negative gradient.

**Example 2**



The gradient of the line segment joining points

*M*(-3;-5) and *N*(1;*t*) is 2¾. Determine the value of *t*.

**Solution:**

Let *M* be the point (*x*1; *y*1) and *N* be the point(*x*2; *y*2).

Then, we get: *x*1= -3 and *y*1= -5 as well as *x*2= 1 and *y*2 = *t*.

=

=

=

But

=

**Task 1**

Determine the gradient of the line segment joining the following pairs of points, by using

the gradient formula:

1.1 A (5; 2) and B (−2; −3)

1.2 P(0; −5) and Q (3; 7)

1.3 E(-1; 3) and F (-1; 5)

1.4 R( -8;0) and S(-2;6)

1.5 A(−*t*; *v*) and B (*4t*; 4*v*), where *v* ≠ 0.

1. In the diagram, AB is parallel to the *x*-axis, BC is parallel to the *y*-axis and D is a point on line AC. The co-ordinates of the points A, B and C are (−2; 3), (3; 3) and (3; −1) respectively.



* 1. Determine the gradient of the horizontal line AB, which is parallel to the *x*-axis.
  2. Determine the gradient of the vertical line BC, which is parallel to the *y*-axis.

2.3 Determine the gradient of the diagonal line AC.

2.4 Write down the gradient of AD. Give a reason for your answer.

1. The gradient of the line segment joining points A(2;-4) and N(-1;) is . Determine the value of .

#### Guided reflection on Activity 2

|  |  |
| --- | --- |
| |  | | --- | | 1. Consider points A (5; 2) and B (−2; −3) on the Cartesian coordinate system. Jerry finds the gradient of AB as follows: = = .   State whether Jerry’s answer is correct or not. If incorrect give a reason?  Incorrect.  should read (-2) –(5) to be consistent with | |

**Answers to Task 1**

1.1 Let A (5; 2) be the point (*x*1; *y*1) and B (−2; −3)be the point(*x*2; *y*2).

Then, we get: *x*1= -5 and *y*1= 2 as well as *x*2= -2 and *y*2 = -3.

We calculate the gradient of line AB, by using the formula = , and making the

appropriate substitutions:

=

=

=

=

1.2 Let P (0; -5) be the point (*x*1; *y*1) and Q (3; 7)be the point(*x*2; *y*2).

Then, we get: *x*1= 0 and *y*1= -5as well as *x*2= 3 and *y*2 = 7.

We calculate the gradient of line PQ, by using the formula = , and making the

appropriate substitutions:

=

=

=

=

1.3 Let E (-1; 3) be the point (*x*1; *y*1) and F (−1; 5)be the point(*x*2; *y*2).

Then, we get: *x*1= -1 and *y*1= 3 as well as *x*2= -1 and *y*2 = 5.

We calculate the gradient of line EF, by using the formula = , and making the

appropriate substitutions:

=

=

=

=

1.4 Let R (-8;0) be the point (*x*1; *y*1) and S (−2; 6)be the point(*x*2; *y*2).

Then, we get: *x*1= -8 and *y*1= 0 as well as *x*2= -2 and *y*2 = 6..

We calculate the gradient of line RS, by using the formula = , and making the

appropriate substitutions:

=

=

=

=

1.5 Let be the point (*x*1; *y*1) and be the point(*x*2; *y*2), where

Then, we get: *x*1= -t and *y*1= v as well as *x*2= 4t and *y*2 = 4v.

We calculate the gradient of line AB, by using the formula = , and making the

appropriate substitutions:

=

=

=

=

2.1Let A (-2;3) be the point (*x*1; *y*1) and B (3; 3)be the point(*x*2; *y*2).

Then, we get: *x*1= -2 and *y*1= 3 as well as *x*2= 3 and *y*2 = 3.

We calculate the gradient of line AB, by using the formula = , and making the

appropriate substitutions:

=

The gradient of a horizontal line is zero.

=

=

=

2.2Let B (3; 3) be the point (*x*1; *y*1) and C (3; -1)be the point(*x*2; *y*2).

Then, we get: *x*1= 3 and *y*1= 3 as well as *x*2= 3 and *y*2 = -1.

We calculate the gradient of line BC, by using the formula = , and making the

appropriate substitutions:

=

=

The gradient of a vertical line is undefined.

=

2.3Let A (-2;3) be the point (*x*1; *y*1) and C (3; -1)be the point(*x*2; *y*2).

Then, we get: *x*1= -2 and *y*1= 3 as well as *x*2= 3 and *y*2 = -1.

We calculate the gradient of line AB, by using the formula = , and making the

appropriate substitutions:

=

=

=

Points A and D lie on AC. Hence the gradient of AC is the same as the gradient of AD.

3. Steps: (a) Find in terms of .

(b) Set up the following equation and solve for : =

Let A (2; 4) be the point (*x*1; *y*1) and N (-1; )be the point(*x*2; *y*2).

Then, we get: *x*1= -2 and *y*1= 4 as well as *x*2= -1and *y*2 = p.

We calculate the gradient of line AN, by using the formula = , and making the

appropriate substitutions:

=

=

=

But =

**Activity 3: The gradient of parallel lines**

**Purpose**:

* Use the Cartesian co-ordinate system to derive formula to calculate the gradient of a  
  line that is parallel to another line.
* Apply the gradient formula for parallel lines

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 45 minutes

**Introduction**

From question 2 in Task 1, you would have observed the following gradient results:

* The gradient of a line parallel to the *x*-axis (a horizontal line) is zero.
* The gradient of a line parallel to the *y*-axis (a vertical line) is undefined.

Now in Investigation 1, we will investigate further about the gradient of parallel lines.

**Investigation 1**

1. Plot the points A, B, C and D on the Cartesian plane and calculate the gradient of AB and CD in each of the cases below:

1.1 A(−1; −2), B(1; 2), C(−1; 2), D(0; 4)

1.2 A(−1; 1), B(−3; 3), C(−2; 3), D(2; −1)

1.3 A(0; 2), B(2; 4), C(0; -2), D(2; 0)

2. Organise your observations from Question 1 into a conjecture or set of conjectures. Write your conjecture(s) using complete sentences.

From Investigation 1, you should have discovered the following:

* For any two non-vertical parallel lines AB and CD, *m*AB = *m*CD .

**Answers to Investigation 1**

* 1. First plot points on Cartesian plane.



Let A(−1; −2) be the point (*x*1; *y*1) and B(1; 2)be the point(*x*2; *y*2).

=

=

=

= 2

Let C(−1; 2) be the point (*x*1; *y*1) and D(0; 4)be the point(*x*2; *y*2).

=

=

=

= 2

1.2



First plot points on Cartesian plane

Let A(−1; 1) be the point (*x*1; *y*1) and B(-3; 3)be the point(*x*2; *y*2).

=

=

=

= -1

Let C(−2; 3) be the point (*x*1; *y*1) and D(2; -1)be the point(*x*2; *y*2).

=

=

=

= -1





Plot points on Cartesian plane

Let A(0; 2) be the point (*x*1; *y*1) and B(2;4)be the point(*x*2; *y*2).

=

=

=

= 1

Let C(0;-2) be the point (*x*1; *y*1) and D(2; 0)be the point(*x*2; *y*2).

=

=

=

= 1

2. For any two non-vertical parallel lines AB and CD, *m*AB = *m*CD .

**Example 1**

Lines AB and CD are defined by the points A(9; 5), B(0; −1), C(0; 5) and D(9; 11). Determine whether lines AB and CD are parallel or not.

**Solution**

= A is(*x*1; *y*1)and B is (*x*2; *y*2)

=

=

=

= C is(*x*1; *y*1)and D is (*x*2; *y*2)

=

=

=

*m*AB = *m*CD =

AB || CD

**Activity 4: Gradient of perpendicular lines**

**Purpose**:

* Use the Cartesian co-ordinate system to derive formula to calculate the gradient of a  
  line that is perpendicular to another line.
* Apply the gradient formula for perpendicular l lines

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 45 minutes

**Introduction**

Now in Investigation 2, we will investigate further about the gradient of parallel lines.

**Investigation 2**

1. In each of the diagrams below, AB is perpendicular to CD.

Calculate the gradient of AB and CD and determine their product.

1.1



1.2

. 

1.3



[Note editor must ensure coordinates are as follows on diagrams

Figure 1.1: A(-2;3), B(3;-2), C(3;3) and D(-2;-2).

Figure 1.2: A(3;3), B(-2;1), C(2;4) and D(4;-1).

Figure 1.3: A(1;4), B(-1;-2), C(-3;0) and D(3;-2)]

2. Organise your observations from Question 1 into a conjecture or set of conjectures. Write your conjecture(s) using complete sentences.

From 1 above, you should have discovered the following:

If AB and CD are non vertical perpendicular lines with gradients *m*AB and *m*CD respectively, then *m*AB × *m*CD = −1.

**Answers to Investigation 2**

1.1



Let A(-2;3) be the point (*x*1; *y*1) and B(3;-2)be the point(*x*2; *y*2).

=

=

=

= -1

Let C(3;3) be the point (*x*1; *y*1) and D(-2;-2).be the point(*x*2; *y*2).

=

=

=

= 1

*m*AB × *m*CD = () x = -1

1.2



Let A(3;3) be the point (*x*1; *y*1) and B(-2;1)be the point(*x*2; *y*2).

=

=

=

=

Let C(2;4) be the point (*x*1; *y*1) and D(4;-1).be the point(*x*2; *y*2).

=

=

=

*m*AB × *m*CD = x = -1

1.3



Let A(1;4) be the point (*x*1; *y*1) and B(-1;-2)be the point(*x*2; *y*2).

=

=

=

=

Let C(-3;0) be the point (*x*1; *y*1) and D(3;-2).be the point(*x*2; *y*2).

=

=

=

=

*m*AB × *m*CD = () x = -1

1. Learners should have discovered and conjectured the following:

If AB and CD are non-vertical perpendicular lines with gradients *m*AB and *m*CD respectively, then *m*AB × *m*CD = −1.

## Example 1



The co-ordinates of four points are A(-3; 4),

B(2;4), C(2;-2) and D(-3;-2). Determine

whether AC is perpendicular to BC or not.

***Solution***

= A is(*x*1; *y*1)and C is (*x*2; *y*2)

=

=

=

= **B** is(*x*1; *y*1)and D is (*x*2; *y*2)

=

=

=

*m*AC × *m*BD = () x = -1

AC is perpendicular to BD

**Activity 5: Gradient and collinear points**

**Purpose**:

* To understand the concept of collinear points
* To show points are collinear

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 15 minutes

**Introduction**

Collinear points are points that lie on the same straight line.

We can show that points A is(*x*A; *y*A)*,* B is (*x*B; *y*B) and C(*x*C; *y*C) are collinear by showing that *m*AB = *m*BC.

Note that a common point B is used in both gradients: *m*AB and *m*BC.

**Example 1**

Show that the points A(-2;-5), B(1;1) and C(3;5) are collinear (lie on a straight line)

**Solution**

For the points to be collinear the gradients of the line segments joining any two points must be equal.

Let be , be .

Then = = =

Let be , be .

Then = = =

The gradient of AB = gradient of BC (i.e.

Therefore A,B and C are collinear.

**TASK 2**

1. Given that PQ || MN, determine the gradient of MN if the gradient of

PQ is:

1.1 1.2 1.3 0 1.4 1.5 0,66 1.6

2. Given that PQ is perpendicular MN, determine the gradient of MN if

the gradient of PQ is:

2.1 3 2.3 2.3 0 2.4 –1 2.5 undefined

3 .In each case, calculate the gradient of AB and CD, and then determine

whether AB and CD are:

i) parallel ii) perpendicular iii) neither perpendicular nor parallel

3.1 A(–5; 2), B(–3; 2) and C(–1; 0), D(–3; 4)

3.4 A( -4; 2), B(-8; 18) and C(–4; 11), D(4; 13)

4. Show that the set of points are collinear:

4.1 A(–3; -4), B(2; 1) and C(4; 3)

4.2 P(–5; -1), Q(0; 1) and R(10; 5)

5. Points T(*k* ;3) and P(-2;1) are given.

5.1 If = , find the value(s) of .

5.2 If TP | QR and = -4, find the value(s) of .

6. P(1; 2), Q(−2; −4), R(4; −1) and S(*x*; 5) are points on a Cartesian plane. Calculate the value of *x*, if:

6.1 PQ is parallel to SR

6.2 SR is perpendicular to PQ.

6.3 Q, R and S are collinear.

7. A(2;-4), B(-2;-2), C(1;4), D (5;2) are the vertices of quadrilateral ABCD



7.1 Prove that BC |CD

7.2 Prove that BC || A.

7.3 Prove that ABCD is a rectangle.

#### Guided reflection on Task 2

1. If AB is parallel to CD, What can you say about their gradients

They are equal (i.e. .

1. If PQ is parallel to MN, What can you say about their gradients

.

1. Write down the gradient of line FG which is parallel to the x-axis.

0

1. Is the gradient of the line MN which is parallel to the y-axis defined or undefined?

Undefined

|  |
| --- |
| . |

**Answers to Task 2**

1.1 1.2 1.3 0 1.4 1.5 0,66 1.6

* 1. PQ ⊥ MN ………… (Given)

* 1. PQ ⊥ MN ………… (Given)

* 1. PQ ⊥ MN ………… (Given)

* 1. PQ ⊥ MN ………… (Given)

2.5 is undefined

⇒ PQ is a vertical line on the y-axis or parallel to the y-axis.

PQ ⊥ MN

MN is either a horizontal line on the x-axis or parallel to the x-axis

the gradient of a horizontal line on the x-axis or parallel to the x-axis is zero

is 0.

3.1 Let A(–5; 2) be and be .

Then = = =

Let be and be .

Then = = =



AB is not parallel to CD

AB is not perpendicular to CD

From (i) and (ii) it is clear that AB and CD are neither perpendicular nor parallel.

3.4 Let A(–4; 2) be and be .

Then = = =

Let be and be .

Then = = =

AB is perpendicular to CD

4.1 For the points to be collinear the gradients of the line segments joining any two points must be equal.

Let be and B(2; 1) be .

Then = = =

Let B(2; 1) be , C(4; 3)be .

Then = = =

The gradient of AB = gradient of BC (i.e.

Therefore A,B and C are collinear.

4.2 For the points to be collinear the gradients of the line segments joining any two points must be equal.

Let P(–5; -1)be and Q(0; 1) be .

Then = = =

Let P(–5; -1) be , and R(10; 5) be .

Then = = =

The gradient of PQ = gradient of PR (i.e.

Therefore P,Q and R are collinear.

5.1 Steps: (a) Find in terms of .

(b) Set up the following equation and solve for : =

Let T(*k* ;3) be the point (*x*1; *y*1) and P(-2;1))be the point(*x*2; *y*2).

Then, we get: *x*1= and *y*1= 3 as well as *x*2= -2and *y*2 = 1.

We calculate the gradient of line TP, by using the formula = , and making the

appropriate substitutions:

=

=

=

but =

5.2 **Solution:**

Steps: (a) Find in terms of .

(b) Use the fact that TP ⊥ QR ⇒ to set up an equation of the form

to solve for *k*.

Let T(*k* ;3) be the point (*x*1; *y*1) and P(-2;1))be the point(*x*2; *y*2).

Then, we get: *x*1= and *y*1= 3 as well as *x*2= -2and *y*2 = 1.

We calculate the gradient of line TP, by using the formula = , and making the

appropriate substitutions:

=

=

=

but TP ⊥ QR … Given

⇒

but ….Given

= -1

=

=

6.1 Let P(1; 2) be and Q(−2; −4) be .

Then = = =

Let be and be .

Then = = =

Now, PQ//SR … (given)

⇒

6.2 We have from 6.1, and = .

Now, SR ⊥ PQ … Given

⇒

= -1

=

For points Q,R and S to be collinear the gradients of the line segments joining any two points must be equal. We use this fact to form an equation and solve for .

6.3 .From 6.1, we = = =

We now find the gradient of QR

Let be and be .

Then = = =

But points P,Q and Rare collinear.

⇒

7.

****

7.1 To prove that BC⊥ CD, we must show that

Let B(–2; -2) be and be .

Then = = =

Let be and be .

Then = = =

BC is perpendicular to CD

7.2 To prove that BC || AD, we must show that =

Let B(–2; -2) be and be .

Then = = =

Let be and be .

Then = = = = 2

BC//AD

7.3 To prove that ABCD is a rectangle, it is necessary and sufficient to show that ABCD is a parallelogram with a right angle.

We have already shown that BC is perpendicular to CD in 8.1, hence we can conclude that

= (i.e. ABCD has a right angle).

All that remains now is to prove that ABCD is a parallelogram. We can prove that ABCD is a parallelogram by showing that both pairs of opposite sides are parallel: AB//DC and BC//AD.

We have already shown in 8.2 that BC//AD using gradients. So, we can similarly show that AB//DC using gradients.

**Proof:**

To prove that AB || DC, we must show that =

Let A(2; -4) be and be .

Then = = =

Let D(5;2) be and be

Then = = = =

AB//DC

Now in Quad ABCD,

AB//DC … proved above

and BC//AD … proved in 8.2

ABCD is a parallelogram … both pairs of opposite sides are parallel.

but = … BC ⊥CD (proved in 8.1)

ABCD is a rectangle … A parallelogram with a right angle.

**Summary Assessment**

1. Consider the points P(-3;2) and Q( 2;6) in the Cartesian plane.
   1. Determine the gradient of PQ.
2. In each case, calculate the gradient of AB and CD, and then determine

whether AB and CD are:

i) parallel ii) perpendicular iii) neither perpendicular nor parallel

2.1 A(–8; 2), B(0; 6) and C(8; –11), D(–2; 4)

* 1. A(–2; –1), B(1; –3) and C(5; 3), D(2; 5)

1. Consider the points A(1;2), B(-2;3), P(-1;-3) and Q(t;-1) in the Cartesian plane. Calculate *t* if:
   1. AB is parallel to PQ. (3)
   2. is perpendicular to PQ. (3)
   3. AB = PQ (4)
2. Collinear pointsare points that lie on the same straight line. If A, B and C are collinear, then *m*AB = *m*BC = *m*AC.

Show that points A (−4; 13), B(2; 1) and C(8; −11) are collinear. (4)

1. A(3;1), B(-4;6) and C(*k*;3) are three points on a straight line. Find the value of *k*. (4)
2. R(-4;1), U(1;4), S(4;2) and T(-1;-1) are vertices of a quadrilateral



* 1. Calculate the gradients of RU, US, ST and TP. (5)
  2. Identify the sides that are parallel or perpendicular in quadrilateral RUST.

Give reason(s) in each case. (4)

* 1. Is RUST a parallelogram? Why or why not? (2)

7. P (-1;1), E(-3;4), R(1;1) and M(3;-2) the vertices of a quadrilateral..

7.1 Make a drawing of the quadrilateral on the Cartesian co-ordinate

system.

7.2 Using the gradient formula, prove that PERM is a parallelogram.

8. A(-3;3), B(1;5), C (4;5) and D(-4;-1)



are the vertices of quadrilateral ABCD.

Prove that ABCD is a trapezium.

#### Guided reflection on Summary Assessment

|  |
| --- |
| 1. Use the answers given for the summary assessment to check if your answers were correct. 2. Did you realize why you got some your answers incorrect? 3. Have you been able to correct errors you made? |

**Answers for summary assessment**

1. Let P(-3;2) be and Q(2;6) be .

Then = = =

2,1 Let A(–8; 2) be and be .

Then = = =

Let be and be .

Then = = =



AB is not parallel to CD

AB is not perpendicular to CD

From (i) and (ii) it is clear that AB and CD are neither perpendicular nor parallel.

2.2 Let A(–2; -1) be and be .

Then = = =

Let be and be .

Then = = =



AB is parallel to CD

3.1AB is parallel to PQ …. (given)

=

Let A(1; 2) be and be .

Then = = =

Let P(-1; 3) be and be .

Then = = =

=

=

=

3.2AB is perpendicular to PQ …. (given)

=

=

3.3 Let A(1;2) be the point(*x*1; *y*1) and B(-2;3) be the point (*x*2; *y*2).

AB =

=

=

=

= units

Let P(-1;-3) be the point(*x*1; *y*1) and Q(t;-1) be the point (*x*2; *y*2).

PQ =

=

=

=

=

But AB = PQ … Given

=

=

=

=

=

4. For the points to be collinear the gradients of the line segments joining any two points must be equal.

Let be and be .

Then = = =

Let be and be .

Then = = =

The gradient of AB = gradient of BC (i.e.

Therefore A,B and C are collinear.

5. Let be and be .

Then = = =

Let be and be .

Then = = =

But A, B and C lie on the straight line (i.e. A,B and C are collinear)

6. R(-4;1), U(1;4), S(4;2) and T(-1;1) are vertices of a quadrilateral



6.1 Let R(-4;1) be and U(1;4), be .

Then = = =

Let U(1;4) be and S(4;2), be .

Then = = =

Let S(4;2) be and T(-1;-1), be .

Then = = =

Let T(-1;-1) be and R(-4;1), be .

Then = = =

6.2

RU//ST

US//TR

6.3 RUST is a parallelogram because both pairs of opposite sides are parallel.

**7.1**



7.2 A quadrilateral is a parallelogram if we can prove that

either both pairs of opposite sides are parallel,

or both pairs of opposite sides are equal,

or both pairs of opposite angles are equal,

or the diagonals bisect each other,

or one side is equal and parallel to the opposite side.

As the question requires us to prove that PERM is a parallelogram using the gradient formula, we will prove that quad PERM is a parallelogram by showing that both pairs of opposite sides of quad PERM are parallel.

By referring to out drawing in 7.1, it makes sense to try and show that PM//ER and PE//MR.

Let P(–1; 1) be and Mbe .

Then = = =

Let be and be .

Then = = =

PM//CD

Let P(–1; 1) be and Ebe .

Then = = =

Let be and be .

Then = = =

PE//MR

Now, in Quad PERM:

PM//CD … proved above

and PE//MR … proved above

PERM is a parallelogram … both pairs of opposite sides are parallel.

8.



To show that ABCD is a trapezium, we must show that one pair opposite sides is parallel. It is useful to work with the given sketch, to try and establish which pair of lines looks more or less parallel and then to try to show that they are indeed parallel. Our Sketch suggests that we should work with AB and DC.

To prove that AB || DC, we must show that =

Let A(-3; 3) be and be .

Then = = =

Let D (-4;1) be and be

Then = = = =

AB//DC

ABCD is a trapezuim … one pair of opposite sides is parallel.

## Unit 3: The midpoint of a line segment

## Learning Outcomes

By the end of this unit, learners should be able to

## Use the Cartesian co-ordinate system to derive and apply a formula to calculate the midpoint of the line segment joining any two points (*x*1; *y*1) and (*x*2; *y*2).

## Activity 1: Developing the Midpoint Formula

**Purpose**:

## To derive and apply a formula to calculate the midpoint of the line segment joining any two points (*x*1; *y*1) and (*x*2; *y*2).

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 30 minutes

**Introduction**

If M is the midpoint of any line segment AB, then it can be found exactly halfway between A and B, as illustrated in the figure below:



In right-angle ∆ABC on the right, the coordinates



of the vertices are A(2;1); B (2;7) and C(8;1).

D1, D3, D2 are the midpoints of sides AB, BC and

AC respectively.

* 1. Write down the coordinates of D1, D2 and D3.
  2. How does the coordinates of D1 compare with those of A and B.
  3. How does the coordinates of D2 compare with those of A and C.

1.4 How does the coordinates of D3 compare with those of B and C.

1.5 The answers to Questions 1.2-143, suggest a way to find the

Coordinates of the midpoint of a line segment. Explain the method.

**Example 1**



In the figure on the right, the coordinates of A and B

are (-3;1) and (2;4) respectively. Calculate the

coordinatesof themidpoint M of the line segment AB.

**Solution**

Let (*x;y*) be the coordinates of point M.

Then the *x* coordinate of point M is equal to  = - ½ , and

the *y* coordinate of point M is equal to  = 2½ .

the coordinates of point M are (- ½; 2½).

Now, from exercise 1 and Example 1, you would have realized that the formula to find the coordinates of the midpoint of a line segment joining two points (*x*1; *y*1) and (*x*2; *y*2)are: . This is known as the **midpoint formula**.

**Example 2**

Let *T*(3;-7) be the point (*x*1; *y*1), *P*( -4; 5) be the point (*x*2; *y*2) and *M*(*x*;*y*) the midpoint of the line segment *TP*.

*M* is the point: =

*M* is the point (- ½; -1)

**Example 3**



*R*(2;-1) is the midpoint of the line segment *PQ*.

Find the coordinates of *Q* when *P* is (6;2).

**Solution**

Let *Q* be the point (*x*1; *y*1), *P*(6;-2) be the point (*x*2; *y*2) and R( 2; -1) be the point (*x*; *y*) .

**TASK 1**

1. Find the coordinates of the M, the midpoint of AB, for each of the following set of endpoints:

1.1 A (0;4) , B(5;12) 1.2 A(-7;4), B(8;-12)

1.3 A(11;0), B(4; -9) 1.4 A(6;-15), B(-5;-3)

1. A(2;4), B(-3;-3) and C(10;-3) are the vertices of ∆ABC.



2.1 Find the coordinates of D, the midpoint of AB.

2.2 Find the coordinates of E, the midpoint of AC.

2.3 Calculate the lengths of DE and BC.

2.4 What can you say about DE and BC?

1. E(-5;2), F (2;2), G(2;-2) and H(-5;-2) are the vertices of rectangle EFGH.



3.1 Find the midpoint of EG.

3.2 Find the midpoint of FH.

3.3 a. Are the midpoints of EG and FH the same point?

b. What property of the diagonals of a rectangle does your response 3.3.a confirms.

1. E(-4;6); F(3;5);G(3;0) and H(-4;1) are the vertices of a quadrilateral.



4.1 Determine the midpoints of each side of the quadrilateral.

4.2 Find the midpoint of diagonal EG.

4.3 Find the midpoint of diagonal HF.

4.4 What can you say about the diagonals EF and HF.

4.5 Is quadrilateral a parallelogram or not? Justify your answer.

1. The point (1;1) is the midpoint of the line joining (-2;-3) and (*a*;*b*). Determine *a* and *b*.
2. P(5;6), Q(6;2), R(-1;-3) and S(-2;1) are the vertices of quadrilateral PQRS. Show that PQRS is a parallelogram by showing that the diagonals PR and SQ bisect each other.

#### Guided reflection on Activity 1

|  |
| --- |
| 1. If A (*x*1; *y*1) and B(*x*2; *y*2)are points on a line, the write down a formula to calculate the midpoint of AB      1. The point (1;1) is the midpoint of the line joining (-2;-3) and (*a*;*b*). Explain how you will determine the values of *a* and *b*.   Let P(-2;-3) be the point (*x*1; *y*1),R(*a*;*b*) be the point (*x*2; *y*2) and Q(1;1) be the point (*x*; *y*).  Then substitute into the following formula and solve for a:  Similarly substitute into the following formula and solve for b: |

**Answers to Task 1**

1.1 Let A (0;4) be the point (*x*1; *y*1),B(5;12) be the point (*x*2; *y*2) and M(*x*; *y*) be the midpoint of point line segment AB.

*M* is the point: =

*M* is the point (-2½; 8)

1.2 Let A (-7; 4) be the point (*x*1; *y*1),B(8;-12) be the point (*x*2; *y*2) and M(*x*; *y*) be the midpoint of point line segment AB.

*M* is the point: =

*M* is the point (-1; -4)

1.3 Let A (11; 0) be the point (*x*1; *y*1),B(4;-9) be the point (*x*2; *y*2) and M(*x*; *y*) be the midpoint of point line segment AB.

*M* is the point: =

*M* is the point (7½; -4½)

1.4 Let A(6;15) be the point (*x*1; *y*1),B(-5;-3) be the point (*x*2; *y*2) and M(*x*; *y*) be the midpoint of point line segment AB.

*M* is the point: =

*M* is the point (½; 6)

2. A(2;4), B(-3;-3) and C(10;-3) are the vertices of ∆ABC.



.

2.1 Let A(2;4) be the point (*x*1; *y*1),B(-3;-3) be the point (*x*2; *y*2) andD(*x*; *y*) be the midpoint of point line segment AB.

*D* is the point: =

*D* is the point (-½; ½ ).

2.2 Let A(2;4) be the point (*x*1; *y*1),C(10;-3) be the point (*x*2; *y*2) and E(*x*; *y*) be the midpoint of point line segment AB.

E is the point: =

E is the point (6; ½ ).

2.3 Let D(-½; ½ )be the point(*x*1; *y*1) and E(6; ½ )be the point (*x*2; *y*2).

DE=

=

=

=

= units

= units

Let B(-3;-3 ) be the point(*x*1; *y*1) and C(10; -3)be the point (*x*2; *y*2).

BC=

=

=

= units

= 13 units

2.4 DE =

Alternatively: Learners by calculating and can show that , and then conclude DE//BC.

1. E(-5;2), F (2;2), G(2;-2) and H(-5;-2) are the vertices of rectangle EFGH.



3.1 Let E(-5;2) be the point (*x*1; *y*1),G(2;-2) be the point (*x*2; *y*2) and M(*x*; *y*) be the midpoint of point line segment AB.

Mis the point: =

M is the point (; 0).

3.2 Let F(2;2) be the point (*x*1; *y*1),H(-5;-2) be the point (*x*2; *y*2) and N(*x*; *y*) be the midpoint of point line segment AB.

Nis the point: =

N is the point (; 0).

3.3 (a) Yes.

(b) The diagonals of a rectangle bisect each other.

1. E(-4;6); F(3;5);G(3;0) and H(-4;1) are the vertices of a quadrilateral.



* 1. Let E(-4;6) be the point (*x*1; *y*1),F(3;5) be the point (*x*2; *y*2) and M(*x*; *y*) be the midpoint of point line segment EF.

Mis the point: =

M is the point (; 5).

Let F(3;5) be the point (*x*1; *y*1),G(3;0) be the point (*x*2; *y*2) and N(*x*; *y*) be the midpoint of point line segment FG.

Nis the point: =

N is the point (; 2).

Let H(-4;1) be the point (*x*1; *y*1),G(3;0) be the point (*x*2; *y*2) and P(*x*; *y*) be the midpoint of point line segment HG.

P is the point: =

P is the point (; ).

Let E(-4;5) be the point (*x*1; *y*1),H(-4;1) be the point (*x*2; *y*2) and Q(*x*; *y*) be the midpoint of point line segment HG.

Q is the point: =

Q is the point (;3).

* 1. Let E(-4;6) be the point (*x*1; *y*1),G(3;0) be the point (*x*2; *y*2) and R(*x*; *y*) be the midpoint of point line segment EF.

Ris the point: =

R is the point (; 3).

* 1. Let H(-4;1) be the point (*x*1; *y*1),F(3;5) be the point (*x*2; *y*2) and S(*x*; *y*) be the midpoint of point line segment EF.

Sis the point: =

S is the point (; 3).

* 1. The diagonals bisect each other.
  2. Yes.

=

EG and HF bisect each other

EFGH is parallelogram. … Diagonals bisect each other

1. Let P(-2;-3) be the point (*x*1; *y*1),R(*a*;*b*) be the point (*x*2; *y*2) and Q(1;1) be the point (*x*; *y*) .

R is the point( 4;5)

1. Let P(5;6) be the point (*x*1; *y*1),R(-1;-3) be the point (*x*2; *y*2) and M(*x*; *y*) be the midpoint of point line segment PR.

Mis the point: =

M is the point (2; ).

Let S(-2;1) be the point (*x*1; *y*1),Q(6;2) be the point (*x*2; *y*2) and N(*x*; *y*) be the midpoint of point line segment SQ.

Nis the point: =

N is the point (2; ).

= (i.e. the diagonals meet in a common midpoint)

Diagonals PR and SQ bisect each other (2; ).

PQRS is parallelogram.

## Unit 4: Equation of a line

## Learning Outcomes

By the end of this unit, learners should be able to

## Use the Cartesian co-ordinate system to derive and apply the equation of a line between two given points;

## Use the Cartesian co-ordinate system to derive and apply the equation of a line parallel or perpendicular to another given line.

## Activity 1: Equation of straight line given the gradient and y-intercept

**Purpose**:

* Determine the equation of a line given the gradient and y-intercept

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 20 minutes

**Introduction**



The equation of a straight line is usually written in the form .

Gradient

(slope) (y-intercept)

(the point where the graph cuts the y-axis when

the x- value is 0)

In the given diagram, the gradient of line AB is and its -intercept is 1.

Hence, the equation of line AB is .

Note you can find the gradient of line AB by using the formula for gradients,

, and the considering the (0;1) as ( and (-2;0) as ,

Then we get:

**Task 1**

1. Determine the gradient and y-intercept for each of the straight line in the table below:

|  |  |  |
| --- | --- | --- |
| Equation | Gradient | y-intercept |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

1. **Determine the equations of the lines given below in the form**
2. gradient 5, y-intercept 2 (b) gradient , y-intercept -4
3. gradient 2, passing through (0;0) (d) gradient , passing through (0;2)

#### Guided reflection on Activity 1

|  |
| --- |
| .   1. Explain how you will determine the gradient and y-intercept of the line 3.   We will rewrite the equation in the form . Then the numerical value of m, which is the coefficient of will give us the gradient and the numerical value of will give us the y-intercept |

**Answers to Task 1**

|  |  |  |
| --- | --- | --- |
| Equation | Gradient | y-intercept |
|  | 2 | 5 |
|  |  | 2 |
|  | 4 | 0 |
|  |  |  |
|  | 3 | 2 |

1. (a)

(b)

(c)

(d)

**Activity 2 Using the gradient and one point**

**Purpose**:

* Determine the equation of a line given the gradient and one point

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 20 minutes

**Introduction**

Suppose that we want to find the equation of a line which has a gradient 2 and passes through the point (-1;3). Here, whilst we know the gradient, we do not know the value of the *y*-intercept *c*.

We start with the general equation of a straight line *y* = *mx* + *c*.

We know the gradient is 2 and so we can substitute this value for into *y* = *mx* + *c.*

This gives .

We now use the fact that the line passes through (-1; 3). This means that when *x* = -1, *y* must be 3. Substituting these values into , we find

So the equation of the line is

We can work out a general formula for problems of this type by using the same method. We shall take a general line with gradient *m*, passing through the fixed point *A*(*x*1*, y*1).

We start with the general equation of a straight line *y* = *mx* + *c*.

We now use the fact that the line passes through *A*(*x*1*, y*1). This means that when *x* = *x*1, *y* must be *y*1. Substituting these values we find

*y*1 = *mx*1 + *c*

*so that c* = *y*1 − *mx*1

So the equation of the line is *y* = *mx* + *y*1 − *mx*1.

We can write this in the alternative form

This then represents a straight line with gradient *m*, passing through the point (*x*1*, y*1). So this general form is useful if you know the gradient and one point on the line. This is called the **point-gradient formula**.

**Example**

Determine the equation of the line passing through A(−2; −6) with a gradient of –7.

**Solution**

Here we have a gradient and one point. So we use our point-gradient formula:

*y* − *y*1 = *m*(*x* − *x*1)

*y* − (−6) = −7(*x* − (−2)) where A is (*x*1; *y*1) and *m* = −7

*y* + 6 = −7*x* + 14

*y* = −7*x* + 8 which is in the form *y = mx + c*

**Task 2**

1. Determine the equation of the lines passing through A and having a gradient *m* (give the equation in the form

(a) A(−4; 2), *m* = −3 (b) A(1; 6), *m* = 2 (c) A(5; −5), *m* = −1/3

(d) A(0; 0), *m* = −3 (e) A(−½; −2½), *m* = −3/2

(f) A(0; −3), *m* = −4/3 (g) A(−3; −2), *m* = 0

#### Guided reflection on Activity 1

|  |
| --- |
| .   1. Explain how you will determine the equation of the lines passing through A(−4; 2) and having a gradient *m* = −3   I will use the formula and let (−4; 2)= ( with *m* = −3.  I will substitute for , 2 for and *m* = −3 in , and simplify to form . |

**Activity 3: Equation of a straight line through two given points**

**Purpose**:

## Use the Cartesian co-ordinate system to derive a formula to calculate the equation a line between two given points

## Apply the two-point formula to calculate the equation a line between 2 points.

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 20 minutes

**Introduction**

What should we do if we want to find the equation of a straight line which passes through the two points (−2*,* 3) and (1*,* 5)?

Here we don’t know the gradient of the line, so it seems as though we cannot use any of the formulae we have found so far. But we do know two points on the line, and so we can use them to work out the gradient.

By using the gradient the formula , and the considering (−2*,* 3) as ( and (1*,* 5) as , then we get: .

In addition, we know two points on the line, so we could choose one of the points say (1*,* 5), and then use the **point-gradient formula**:

Looking back, when we are given a line passing through two points, we can take the point gradient formula, and replace to give us a general formula:

.

The above formula, which is called the **two-point formula**, can be used to determine the equation of a line that passes through given two points.

**Task 3**

Determine the equations of the straight lines through the following points:

1. (3;-5) and (-1;3)
2. (4;7) and (-2;2)
3. (-1;2) and (-3;5)

**Guided reflection on Activity 3**

|  |
| --- |
| Explain how you will determine the equation of the lines passing through points A(3;-5) and B(-1;3),  I will first find the , and then use the formula where I will let = ( with *m* = . I will substitute  I will substitute for , -5 for and *m* = in , and simplify to form .  Note: Equivalently we can do the same by working directly with the two-point formula, |

**Answers to Task 3**

1. By using the gradient the formula , and the considering (3;-5) as ( and (-1;3) as , then we get: .

In addition, we know two points on the line, so we could choose one of the points say (3;-5), and then use the **point-gradient formula**:

1. Consider (4;7) as ( and (-2;2) as , and substitute respectively into the two-point formula

1. Consider (-1;2) as ( and (-3;5) as and substitute respectively into the two-point formula

**Activity 4 Equation of a line through one point and perpendicular to a given line**

**Purpose**:

* Develop and apply a procedure to calculate the equation a line passing through one point and perpendicular to another given line

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 25 minutes

**Example**



Determine the equation of a line PQ, which passes through P(–3; 9) and is perpendicular to the line CD defined by 4*x* – *y* = 2.

**Solution**

Step1: Determine *m*CD by writing 4*x* – *y* = 2 in the form *y* = *mx* + *c*.

4*x* –*y* = 2

*y* = 4*x* –2

Therefore *m*CD = 4

Step 2: Use the fact that PQ is perpendicular to CD to find *m*PQ:

*m*PQ × *m*CD = –1 (Given PQ is perpendicular to CD)

Therefore*m*PQ × 4 = –1

Therefore*m*PQ  = –1/4

Step3: Use the point gradient formula to find the equation of line CD

*y* – *y1 = m(x* – *x1)*

where A is (*x*1; *y*1) and *m* = −1/4

(which is in the form *y =mx +c*)

**Task 4**

1. Determine the equation of line AB which passes through (1; 1) and is perpendicular to line CD defined by
2. Determine the equation of line PQ which passes through (–3; 1) and is perpendicular to the line defined by .

**Guided reflection on Activity 4**

|  |
| --- |
| 1. Write down the steps you will use to calculate the equation of line AB which passes through (1; 1) and is perpendicular to line CD defined by   Step 1: Determine the gradient ( of line CD defined  Step 2: Use the fact that AB is perpendicular to CD to calculate  i.e. used  Step3: Let (1; 1)= (, use as the value for , and substitute for into the **point-gradient formula**: |

**Answers to Task 4**

AB is perpendicular to CD

Let (1; 1)= (, use as the value for , and substitute for into the **point-gradient formula**:

1. Write in the form

**….. which is the equation of line CD**

AB is perpendicular to CD

Let (–3; 1)= (, use as the value for , and substitute for into the **point-gradient formula**:

**Activity 5 Equation of a line through one point and parallel to a given line**

**Purpose**:

* Develop and apply a procedure to calculate the equation a line passing through one point and parallel to another given line

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 25 minutes

**Example**



The equation of a line AB is 3*y* – 4*x* +12 = 0.

Determine the equation of the line CD passing through M(−1; 3) and parallel to line AB.

***Solution***

Step 1: To find *m*AB, write 3*y* – 4*x* +12 = 0 in the form *y = mx +c.*

3*y* – 4*x* +12 = 0

3*y* = 4*x* –12

*y* = [4/3]*x* − 4

Therefore*m*AB = [4/3]

Step 2: To determine *m*CD, use the fact that CD || AB:

*m*CD = *m*AB = [4/3] (Given CD || AB.)

Step 3: Use the point-gradient formula to find the equation of line CD:

*y* – *y1 = m(x* – *x1)*

*y* – (3) = 4/3(*x* – (−1)) where A is (*x*1; *y*1) and *m* = 4/3

*y* – 3 = 4/3*x* + 4/3

*y* = 4/3 *x* + 4 1/3 which is in the form *y = mx + c*

**Task 5**

1. Determine the equation of line AB which passes through (–3; 1) and parallel to line CD defined .
2. Determine the equation of line MN which passes through (2; ) and parallel to line PQ defined by .

**Guided reflection on Activity 5**

|  |
| --- |
| Write down the steps you would follow to determine the equation of line AB which passes through (–3; 1) and parallel to line CD defined by .  Step 1: Write in the form  Step 2: Write down the gradient of line CD.  Step 3: Use the fact that when 2 lines are parallel, their gradients are equal to determine the gradient of line AB . In this case  Step 4: Let = ( with *m* = .  Substitute for , 1 for and *m* = in , and simplify to form . |

**Answers to Task 5**

1. .

…………… Equation of line CD in the form

AB is parallel to CD

Let (-3; 1)= (, use as the value for , and substitute for into the **point-gradient formula**:

1. .

…………… Equation of line PQ in the form

MN is parallel to PQ

Let (2; )= (, use as the value for , and substitute for into the **point-gradient formula**:

**Summary Assessment**

1. Determine the equations of the straight line passing through (2;-4) and (-3;6)
2. Determine the equation of line AB which passes through (–4; 3) and parallel to line CD defined
3. Determine the equation of line AB which passes through (2; ) and is perpendicular to line CD defined by

#### Guided reflection on Summary Assessment

|  |
| --- |
| 1. Use the answers given for the summary assessment to check if your answers were correct. 2. Did you realize why you got some your answers incorrect? 3. Have you been able to correct errors you made? |

**Answers for summary assessment**

1. Consider (2; -4) as ( and (-3;6) as , and substitute respectively into the two-point formula



…………… Equation of line CD in the form

AB is parallel to CD

Let (-4;3) = (, use as the value for , and substitute for into the **point-gradient formula**:

AB is perpendicular to CD

Let= (, use as the value for , and substitute for into the **point-gradient formula**:

## Unit 5: Inclination of a line

## Learning Outcomes

By the end of this unit, learners should be able to

## Use the Cartesian co-ordinate system to derive and apply the angle of inclination of a line (i.e. tan *Ө*= *m*AB)

## Activity 1: Inclination of a line

**Purpose**:

* Discover that gradient of a line is equal to the tangent of the angle that the line makes with the.
* Determine the inclination of a line when the gradient ins known or given
* Determine the equation of a line when given 2 points on the line
* Determine the gradient of a line when the inlination is given

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 45 minutes



In given figure, A(*x*2; *y*2) and B(*x*1; *y*1) are two points on the Cartesian plane. The straight line ABD makes an angle *Ө* with the positive *x*-axis. BC is parallel to the *y*-axis and AC is parallel to the *x*-axis.

a) What is the measure of ?

b) Write down the co-ordinates of C.

c) What is the length of AB in terms of *x1* and *x2*?

d) What is the length of BC in terms of *y1* and *y*2?

e) Why is ?

f) Copy and complete the following:

= *m*AB

g) Why is *m*AD = *m*AB = tan *Ө*?

In the above activity, you would have discovered that the gradient of AB is the same as the gradient of line ABD (or line AD). This means that the gradient of AB is the same as the tangent of the angle that it makes with positive *x*-axis i.e. *m*AB = tan *Ө.*

The angle *Ө* between the line AB and the positive direction of the *x*-axis measured in an anti-clockwise direction is called the **angle of inclination**, or sometimes just the inclination of line AB.

Now for line segment AB, we have tan *Ө* = *m*AB .

So *Ө =* tan-1 (*m*AB), where 0º <*Ө<*180º (since if *Ө is* 180º this would be the same angle of inclination of 0º).

**Example 1:**

****

The gradient of AB is . Determine the inclination of line AB.

**Solution:**

tan *Ө* = *m*AB *=*

U*se your calculator as follows:*

* *Press Shift and then press tan button to get to*
* *Key in*
* *Close the bracket*
* *Press the equal sign and you will get 33, 690006753 showing on your calculator. This is called your reference angle.*
* *Round 33, 690006753 to two decimal places to get*

*Since the slope of the line is positive, the angle of inclination of AB will actually be equal to angle of reference*

*Therefore the inclination of AB is*

**Example 2**

****

The gradient of AB is - . Determine the inclination of line TP.

**Solution:**

U*se your calculator as follows:*

* *Press Shift and then press tan button to get to*
* *Key in (-*
* *Close the bracket*
* *Press the equal sign and you will get -18,43494882 showing on your calculator. This value is negative.*
* *We can see from diagram that the size of is an obtuse angle. To get the size of , you should add to -18,43494882. You will then get 161.5650512 showing on your calculator.*
* *Round 161.5650512 to two decimal places to get .*

*Since the slope of the line is negative, the angle of inclination of TP will actually be equal to this obtuse angle of*

*Therefore the inclination of TP is .*

**Example 3**



Points A(1; −3) and B(−4; 3) are points on the

Cartesian plane. Determine the size of *Ө*, the angle of inclination of AB, correct to one decimal place.

**Solution**

tan *Ө* = *m*AB

A is (*x*1; *y*1) and B is (*x*2; *y*2)

=

Using a calculator, correct to one decimal place. The slope of the line is positive *, the angle of inclination of AB will actually be equal to angle of reference*

*Therefore the inclination of AB is*

**Note:** + *k*.180º, *k ε Z,*

*But the* angle of inclination is never greater than 180º *(i.e.*  0º < *Ө <*180º)

**Example 4**

Determine the angle of inclination of the line , correct to one decimal place.

**Solution**

Let the angle of inclination be *Ө.*

Write the equation in standard form:

so, *m* = −8/3

But tan *Ө* = *m*

Therefore tan *Ө* = −8/3

Using a calculator, *Ө=* −69.4 º + *k*.180º, *k ε Z,*

and since 0 º <*Ө<*180 º,

*Ө =* 110,6º

**Example 5**

The angle of inclination of line AB with respect to the -axis is . Determine the gradient of line AB.

Press tan on your calculator

Key in 60

Close bracket

Press ‘=” sign

You will get

**Solution**

**TASK 1**



The gradient of AB is . Determine the inclination of line AB.

2. The gradient of TP is . Determine the inclination of line TP.

1. Calculate the inclination of line AB passing through points A and B:

(a) A (4; 3) and B(6; 10) (b) A(−2; 0) and B(−4; 3)

4. Calculate the inclination of the following straight lines: 

a) *y* + 3*x* =2 b) 3*y* = −*x* + 12

1. The angle of inclination of line TP with respect to the -axis is. Determine the gradient of line TP.

6.



In the figure above, the equation of line AB is and the equation of line CD is

a) Determine *Ө*1, the angle of inclination of line AB.

b) Determine*Ө*2, the angle of inclination of line CD.

c) Hence, or otherwise determine the size of β, the angle between line AB and line CD.

#### Guided reflection on Activity 1

|  |
| --- |
| 1. When given the gradient of line AB to be , explain how you will find the inclination of AB.   Let be the angle of inclination. Then set . Then determine  to get the value of   1. If , do you think that will be more than ?     As the gradient is negative, TP forms and obtuse angle with the axis (in anti-clockwise direction). Hence will be more than but less than .   1. State the exterior angle of a triangle theorem.   The exterior angle of a triangle is equal to the sum of its interior opposite angle. |

**Answers to Task 1**

1**.**

Inclination of TP =

*(when rounded to 2 decimal places)*

2.

Inclination of TP =

*(when rounded to 2 decimal places)*

3. (a) A (4; 3) and B(6; 10)

*=* A is (*x*1; *y*1) and B is (*x*2; *y*2)

Using a calculator, correct to 2 decimal places. Since the slope of the line is positive *the angle of inclination of AB will actually be equal to angle of reference*

*Therefore the inclination of AB is .*

3. (b) A(−2; 0) and B(−4; 3)

*=* A is (*x*1; *y*1) and B is (*x*2; *y*2)

Inclination of TP =

*(when rounded to 2 decimal places)*

1. *(a)*

*Inclination of line*

1. b)

*Inclination of line* =

=

(rounded to 2 decimal places)

1. (a) = 1

Since the slope of the line is positive *the angle of inclination of AB will actually be equal to angle of reference.*

(b) =

Since the slope of the line is positive *the angle of inclination of AB will actually be equal to angle of reference.*

(c) (exterior angle of a

**Summary Assessment**



****

The gradient of MN is . Determine the inclination of line MN.

1. Calculate the inclination of line AB passing through points A and B:

(a) (-1;-3) and B (6;5) (b) (-5; 2) and B ( -2; -1)

1. Calculate the inclination straight line .
2. The angle of inclination of line CD with respect to the -axis is.

Determine the gradient of line CD.

1. Find the angle between lines AB and CD, passing through the points A(2; 3), B (5; 2) and C(1; 1), D(4; 6)

#### Guided reflection on Summary Assessment

|  |
| --- |
| 1. Use the answers given for the summary assessment to check if your answers were correct. 2. Did you realize why you got some your answers incorrect? 3. Have you been able to correct errors you made? |

**Answers for summary assessment**

1.

Inclination of MN =

*(when rounded to 2 decimal places)*

1. (a)

(-1;-3) and B (6;5)

*=* A is (*x*1; *y*1) and B is (*x*2; *y*2)

Inclination of AB=  *(when rounded to 2 decimal places)*

(b) A(-5; 2) and B ( -2; -1)

*=* A is (*x*1; *y*1) and B is (*x*2; *y*2)

Inclination of TP =



*Inclination of line* =

=

4.

(rounded to 2 decimal places)



A(2; 3), B (5; 2)

Let represent the angle between AB and the axis

A is (*x*1; *y*1) and B is (*x*2; *y*2)

C(1; 1), D(4; 6)

Let represent the angle between CD and the axis

= C is (*x*1; *y*1) and D is (*x*2; *y*2)

=

Since the slope of the line is positive *the angle of inclination of AB will actually be equal to angle of reference.*

Let represent that angle between the lines AB and CD

(exterior angle of a )

## Unit 6: Equation of a circle

## Learning Outcomes

By the end of this unit, learners should be able to

* Use the Cartesian co-ordinate system to derive and apply the equation of a circle (any centre)

## Activity 1: Equation of a circle with centre at origin

**Purpose**: To derive and apply the equation of a circle (any centre)

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 45 minutes

**Introduction**

In this activity, we discuss how to find the equation of a circle centre passing through the origin.



In the diagram above, you are given a circle with centre O (0;0), and radius r .

is any point on the circumference of the circle. Furthermore, AB ┴ AX and OA is joined.

1. Express OB in terms of x.
2. Express BA in terms of y.
3. Express OA in terms of r
4. (a)Express OA2 in terms of OB and BA. Which Theorem did you use in this case.

(b) Using 4(a), write an equation that shows relationship between

From Investigation 1, you would have shown that x2 + y2 = r2 . This equation

x2 + y2 = r2, is the equation of the circle that has the origin at centre.

We can use the distance formula to verify that the equation of circle with centre at origin is as follows:

Let’s consider the origin (0;0) as () and as (). In the case the distance is given by r. The according to the distance formula

**Example 1**

Determine the equation of a circle with centre at the origin (0;0) and a radius of 3 units.

**Solution**

(substitute 3 for *r*)

Example 2

The equation of a circle with centre at origin (0;0) is . What is the radius of the circle?

**Solution:**

Example 3

Determine whether the point (-2; 3) is on the circle

Solution

If (-2;3) is on the circle, it will satisfy the equation.

We have

LHS =

= (-2)2 + (3)2

= 4 + 9

= 13

= RHS

Thus (-2; 3) is on the circle

**Task 1**

1. Determine the equation of a circle with centre at the origin (0;0) and a radius of 7 units.
2. The equation of a circle with centre at origin (0;0) is . What is the radius of the circle?
3. Determine whether the point (1; 2) is on the circle

#### Guided reflection on Activity 1

|  |
| --- |
| 1. Explain how will you determine the equation of a circle with centre at the origin (0;0) and a radius of 7 units.   Replace by 7 in the equation and simplify.   1. How will you show that (1; 2) is on the circle   Substitute x =2 and y =2 into and if it simplifies to 8 then it means the LHS = RHS, and you can then conclude that (1; 2) is on the circle . |

**Answers to Task 1**

(substitute 7 for *r*)

1. If is on the circle, it must satisfy the equation

LHS =

= (1)2 + (2)2

= 1 + 4

= =5

= RHS

Thus (1;2) does not lie on the circle.

## Activity 2: Equation of a circle with centre not at origin

**Purpose**: Derive and apply the equation of a circle (any centre)

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 45 minutes

**Introduction**

In this activity, we discuss how to find the equation of a circle with any centre passing through a given point



Let the centre of the circle be at point and point be any point on the circumference of the circle with radius . According to the distance formula:

Therefore the equation of the circle with centres and radius r is:

To find the equation of a circle, we must first know the centre and the radius of the circle.

# Example 1

Find the centre and radius of the circle with equation **(x + 1)2 + (y + 3)2  = 25**

**Solution**

Compare the **centre-radius equation** to the given equation.

**(x-a)2 + (y -b)2  = r2** ,

**(x + 1)2 + (y + 3)2  = 25**

a must be –1 to get (x +1).

b must be –3 to get (y + 3).

The centre is (a;b) so it is (-1;-3).

r2 = 25, so the radius is 5 (r is a length, so r cannot be negative).

**Example 2**

Write an equation for the circle with center and radius 7.

**Solution**

What are the two things we need to write an equation of a circle.

Here and ) = . So a = 1 and

Substitute the values of a, b and r into the equation for a circle that has any point as centre.

**(x-1)2 + (y – (-4))2  = 72**

Simplify: (**x-1)2 + (y + 4)2  = 49** is the desired equation in **centre radius form.**

**Example 3**

We may simplify (x-1)2 + (y +4)2  = 49 , which is in centre radius form, further as illustrated below:

(x-1)2 + (y +4)2  = 49

x2 – 2x + 1 + y2  + 8y + 16 = 49

x2 + y2  – 2x + 8y -32 = 0 , which is in the general circle equation form.

**Note:**

1. When the centre – radius equation (x-a)2 + (y -b)2 = r2 is simplified to the form

x2 + y2  + 2dx + 2ey + f = 0, it is known as the general circle equation.

2. In the general circle equation

1. the coefficient of x2 and y2 are equal to 1, and
2. there is no term in

If your equation is in standard form it will make it easier for you to identify the centre and radius. If your equation is in the general form you will need to convert it into the standard form:

Step1: Make sure that the coefficients on the x2 and y2 terms are equal to 1. If not, then divide both sides by the coefficients of the x2 and y2 .

Do you remember how to complete the square

Step2: Complete the square for both x and y.

Step3: Identify the centre and radius of the circle.

# Example 4

Determine the centre and radius of the circle defined by x2 + y2  + 6x + 4y -12 = 0.

**Solution:**

We have x2 + 6x + \_\_ + y2  + 4y + \_\_ = 12 . Leave spaces for completion of the square.

We want to make x2 + 6x into a square. To do this we add the square of half the coefficient of x, that is, (6/2)2 The same applies to y2  + 4y , we add (4/2)2.

=> (x2 + 6x + 9) + (y2  + 4y + 4) = 12 +9 + 4

=> (x + 3)2 + ( y + 2)2 = 25.

The given equation represents a circle with centre (-3;-2) and radius

**Example 5**

Determine the equation of the circle passing through the points A(2;3) and B(7;12), where AB is the diameter.

**Solution:**

The centre of the circle is at the midpoint of the diameter and the radius = ½ diameter

So centre = = (4.5; 7.5) and

So the equation is

**Task 2**

Can I show that a point lies on a circle with any centre?

1. Is each of the following points on the circle with centre (3;2)

and radius 10?

(a) (13;2) (b) (3;-8)

2. For each circles, determine

**a**. the centre **b.** the radius **c.** one point on the circle with the given equation

* 1. (x-5)2 + (y –11)2 = 81
  2. (x + 2)2 + y 2 = 5
  3. (x + 7)2 + (y + 3)2 = 1
  4. x2 + y2  -10x - 4y + 28 = 0.

1. In each question the first point is the centre of the circle. The second point is on the circumference of the circle. Write down the general circle equation for each circle.

(a) (2;3) and (4;-3) (b) (-3; -2) and ( 1,5; 5)

1. Determine the equation of the circle that has the line segment joining points

P( -4; 3) and M( 2; -2) as diameter.

#### Guided reflection on Activity 2

|  |
| --- |
| .   1. Given. Explain how you will determine the centre and radius of the circle.   We re-write in the form =  So then we get (  Then we see that and and  This means that centre is ) and the radius is =1   1. Explain how you will determine the equation of the circle that has the line segment joining points P( -4; 3) and M( 2; -2) as diameter.   Find the midpoint of PM, and use it as the centre of the circle.  Then determine the radius by finding the length of PM and halving it or by finding the distance from the centre to either point P or M.  Then consider the = , and replace by the -coordinate of the  midpoint of PM, by the -coordinate of the midpoint of PM, and replace by the  value of half the length of PM (or distance from the centre to either point P or M) |

**Answers to Task 2**

1. The equation of the circle is (*x*-3)2 + (*y* -2)2 = 100

1. LHS = (13-3)2 + (2 -2)2  = 100 + 0

= 100

= RHS

The point (3; 2) is on the circle.

1. LHS = (3-3)2 + (-8 -2)2

= 0 + 100

= 100

= RHS

The point (3; -8) is on the circle

2.1 . (a) (5; 11)

(b) 9

(c) (-4; 11)

Consider other points that satisfy the circle equation.

* 1. (a) (-2; 0)

(b) 5

(c) (-7; 0) or (3; 0), etc.

Consider other points that satisfy the circle equation.

2.3 (a) (-3; -7)

(b) 1

(c) (-7; -4).

Consider other points that satisfy the circle equation.

2.4 (a) (5; 2)

(b) 1

(c) (5; 1) .

Consider other points that satisfy the circle equation.

3. (a) [4- (-2)]2 + [-3 – 3]2 = *r*2

4 + 36 = *r*2

40 = *r*2

General circle equation: (*x*-2)2 + (*y*-3)2 = 40

*x*2 – 4*x* + 4 + *y*2 – 6*y* + 9 = 40

*x*2 + *y*2 – 4*x*– 6*y* -23 = 0

(b) [1.5 - (-3)]2 + 5 – (-2)]2 = *r*2

20,25 + 64 = *r*2

84,25 = *r*2

General circle equation: (*x* + 3)2 + (*y* + 2)2 = 84 ¼ √

*x*2 + 6*x* + 9 + *y*2 + 4*y* + 4 = 84 ¼

*x*2 + *y*2 + 6*x* + 4*y* -71¼ = 0

4*x*2 + 4*y*2 + 24*x* + 16*y* - 285 = 0

4. Centre of circle: (1; ½)

[-4 -1]2 + [3 - ½ ]2 = *r*2

[-4 -1]2 + 25 + 25/4 = *r*2

125/4 = *r*2

Circle equation: (*x* -1)2 + (*y* - ½)2 = 125/4

**Summary Assessment**

1. Is each of the following points on the circle with centre (3;2)

and radius 10?

* 1. (8;7) 1.2 (9;-6)

2. For each circles, determine

**a**. the centre **b.** the radius **c.** one point on the circle with the given equation

* 1. x2 + y2 = 16
  2. x2 –2x + y2 + 14y + 45 = 0.

1. In each question the first point is the centre of the circle. The second point is on the circumference of the circle. Write down the general circle equation for each circle.

* 1. (-1;-2) and ( -1;0) 3.2 (-5;-3) and ( 0;0)

4, Determine the equation of the circle that touches the y-axis and has the point

(-4; -2) as centre.

#### Guided reflection on Summary Assessment

|  |
| --- |
| 1. Use the answers given for the summary assessment to check if your answers were correct. 2. Did you realize why you got some your answers incorrect? 3. Have you been able to correct errors you made? |

**Answers for summary assessment**

1.

* 1. LHS = (8-3)2 + (7 -2)2

= 25 + 25

= 50

≠ RHS

The point (8; 7) is not on the circle

* 1. LHS = (9-3)2 + (-6 -2)2

= 36 + 64

= 100

= RHS

The point (9; -6) is on the circle

* 1. (a) (0; 0)

(b) 4

(c) (0; -4) or (4; 0), etc.

Consider other points that satisfy the circle equation.

2.2 . (a) (1; -7)

(b) 5

(c) (0; -5).

Consider other points that satisfy the circle equation.

3.1 [1-(-1)]2 + [0 –(-2)]2 = *r*2

0 + 4 = *r*2

4= *r*2

General circle equation: (*x* + 1)2 + (*y* + 2)2 = 4

*x*2 + 2*x* + 1 + *y*2 + 4*y* + 4 = 4

*x*2 + *y*2 + 2*x* + 4*y* + 1 = 0

3.2 (0 + 5]2 + [0 + 3]2 = *r*2

25 + 9 = *r*2

34 = *r*2

General circle equation: (*x* + 5)2 + (*y* + 3)2 = 34

*x*2 + 10*x* + 25 + *y*2 + 6*y* + 9 = 40

*x*2 + *y*2 + 10*x* + 6*y* = 0

4. When the circle touches the *y*-axis, then *r* = h and the equation becomes

(*x* + 4)2 + (*y* + 2)2 = (-4)2

*x*2 + 4*x* + 16 + *y*2 + 4*y* + 4 = 16

*x*2 + *y*2 + 8*x* + 4*y* + 4 = 0

## Unit 7: Equation of a tangent to a circle

## Learning Outcomes

By the end of this unit, learners should be able to

## Use the Cartesian co-ordinate system to derive and apply the equation of a tangent to a circle given a point on the circle.

## Activity 1: Calculating equations of tangents to circles

**Introduction**

**Purpose**: To calculate the equation of a tangent to a circle with centre at: (a) the origin and (b) not at the origin

**Resources**: A pen or pencil, eraser, calculator and paper

**Suggested time:** 45 minutes

In Euclidean geometry we have had lots of experiences with tangents to circles.

The tangent to a circle is a line that touches the circle at one point only and is perpendicular to the radius at the point of contact.

In this section of coordinate geometry, we will focus on how to find the equation of a tangent to a circle at a point on the circle.

# Example 1

Calculate the equation of the tangent to a circlex 2 + y 2 = 29 at the point (2;5) which is on the circle.

****

**Step 1:** Determine the gradient of the OP (i.e. the gradient of the radius from the

centre to the given point):

Let (x1 ;y1) = (0;0) and (x2 ;y2) = (2;5)

=

Can I remember how to find the gradient of a line that is perpendicular to another line?

=

**Step 2:** The tangent is perpendicular to the radius, so find the gradient of the

tangent, using the fact that the product of the gradients of perpendicular lines

is –1.

mOP X mPT = -1

* x mPT = -1
* mPT = -1 X =

**Step 3:** Now you know the gradient of the tangent and a point on it, so you can find the equation, by using y – y1 = m ( x –x1 ) , where (x1 ; y1)= (2;5).

Do I know how to find the equation of a line when I know it’s gradient an a point that lies on it? definition?

y – y1 = m ( x –x1 )

* y – 5 = ( x – 2)
* y –5 = +
* y = x +5
* y =- x + 6 , which is the equation of the tangent PT

# Example 2

Determine the equation of the tangent tox2 + y2  + 2x - 12y + 32 = 0 at the point (-2;8) which is on the circle.

# Solution

**Step1:** First write the equation in the centre- radius form

x2 + y2  + 2x - 12y + 32 = 0

x2 + y2  + 2x - 12y = - 32

=> ( x2 + + 2x + 1) + ( y2  -12y + 36) = -32 +1 + 36

=> (x + 1)2 + ( y - 6)2 = 5.

The given equation represents a circle with centre (-1; 6) and radius equal to √5.



**Step 2**: Find the gradient of MP (radius):

Let (x1 ;y1) = (-1;6) and (x2 ;y2) = (-2;8)

**Step 2:** The tangent is perpendicular to the radius, so find the gradient of the tangent, using the fact that the product of the gradients of perpendicular lines is –1.

mOP X mPT =

* -2x mPT =
* mPT = X ( =

**Step 3:** Now you know the gradient of the tangent and a point on it, so you can find the equation, by using y – y1 = m ( x –x1 ) , where (x1 ;y1)= (-2;8).

y – y1 = m ( x –x1 )

, which is the equation of the tangent PT

**Task 1**

1. Determine the equation of the tangent to a circlex 2 + y 2 = 34 at the point

(3;-5) which is on the circle.

1. Determine the equation of the tangent tox2 + y2  + 6x - 2y -15 = 0 at the point

(-1;4) which is on the circle.

1. Determine the equation of the tangent to the given circleat the given point:

(a) (x - 2)2 + ( y - 5)2 = 13 at (-1; 3)

(b) x2 + y2  -2x +12y + 19 = 0 at (4;-3)

4. (-1;m) and ( 3;n) are points on the circle (x - 1)2 + ( y + 2)2 = 8.

1. Calculate the value(s) of m and n.
2. Determine the equations of the tangents at these points.
3. Determine the angle between these tangents.

.

#### Guided reflection on Activity 1

|  |
| --- |
| 1. Write down the steps to calculate the equation of tangent to circle x2 + y2  + 6x - 2y -15 = 0 at the point (-1;4) which is on the circle.   **Step1:** First write the equation in the centre- radius form.    Let M represent the centre of the circle and P(-1;4) . Write down the coordinates of M after writing down the equation of the circle in the centre- radius form.  **Step 2** Find the gradient of MP (radius)  **Step 3** The tangent is perpendicular to the radius, therefore find the gradient of the tangent, using the fact that the product of the gradients of perpendicular lines is –1.  **Step 4** Now that the gradient of the tangent and a point on it is known, the equation can be found by using: *y* – *y*1 = *m(x* –*x*1), where (*x*1; *y*1) = (-1; 4). |

**Answers to Task 1**

1.

****

**Step 1** Determine the gradient of the OP (i.e. the gradient of the radius from the centre to the given point):

Let (*x*2; *y*2) = (0; 0) and (*x*1; *y*1) = (3; -5)

*m*OP = *y*2 - *y*1

*x*2 - *x*1

= 0-(-5)

0 - 3

= - 5/3

**Step 2** The tangent is perpendicular to the radius, therefore find the gradient of the tangent, using the fact that the product of the gradients of perpendicular lines is –1.

*m*OP *X**m*PT = -1

* -5/3 *x* mPT = -1
* mPT = -1 *x* - 3/5 = 3/5

**Step 4** Now that the gradient of the tangent and a point on it is known, the equation can be found by using: *y* – *y*1 = *m*(*x* –*x*1), where (*x*1; *y*1) = (3; -5).

*y* – *y*1 = *m*(*x* –*x*1)

* *y* – (-5) = 3/5 (*x* – 3)
* *y* + 5 = 3/5 *x* – 9/5
* *y* = 3/5 *x* – 9/5 – 5
* *y* = 3/5 *x* – 6 4/5

# 2. Step 1 First write the equation in the centre- radius form.

*x*2 + *y*2 + 6*x* - 2*y* -15 = 0

*x*2 + 6*x* + 9 + *y*2  - 2*y* + 1 = 15 + 9 + 1

= > (*x* + 3)2 + (*y* - 1)2 = 25.

The given equation represents a circle with centre (-3; 1) and radius equal to 5.



**Step 2** Find the gradient of MP (radius):

Let (*x*1; *y*1) = (-1; 4) and (*x*2; *y*2) = (-3; 1)

*m*MP = *y*2 - *y*1

*x*2 - *x*1

= 1-4\_\_\_

-3- (-1)

= -3/-2

= 3/2

**Step 3** The tangent is perpendicular to the radius, therefore find the gradient of the tangent, using the fact that the product of the gradients of perpendicular lines is –1.

*m*MP *X**m*PT = -1

* 3/2*xm*PT = -1
* *m*PT = -1*x*2/3 = -2/3

**Step 4** Now that the gradient of the tangent and a point on it is known, the equation can be found by using: *y* – *y*1 = *m(x* –*x*1), where (*x*1; *y*1) = (-1; 4).

*y* – *y*1 = *m*(*x* –*x*1)

* *y* – 4 = -2/3(*x* – (-1))
* *y* – 4 = -2/3(*x* + 1)
* *y* –4 = -2/3 *x* – 2/3
* *y* = -2/3 *x* – 2/3 + 4
* *y* = -2/3 *x* + 3 1/3, which is the equation of the tangent PT √√

3. (a) *y* = -3/2 *x* + 3/2

1. *y* = 5/9*x* – 5 2/9

4. (a) (-1-1)2 + (*m* + 2)2 = 8

4 + *m*2 + 4*m* + 4 = 8

*m*2 + 4*m* = 0

*m*(*m* + 4) = 0

*m* = 0 or *m* = -4

(-1-1)2 + (*n +* 2)2 = 8

4 + *n*2 + 4*n* + 4 = 8

*n*2 + 4*n* = 0

*n*(*n* + 4) = 0

*n* = 0 or *n* = -4



The equation of the tangent FG at (-1; 0):

Determine the gradient of AM (i.e. the gradient of the radius from the centre to the given point):

Let (*x*1; *y*1) = (-1; 0) and (*x*2; *y*2) = (1; -2)

*m*AM = *y*2 - *y*1

*x*2 - *x*1

= -2-0\_\_

1- (-1)

= -1

**Step 2** The tangent is perpendicular to the radius, therefore find the gradient of the tangent, using the fact that the product of the gradients of perpendicular lines is –1.

*m*AM ×*m*FG = -1

* -1 × *m*FG = -1
* *m*FG = 1

**Step 3** Now that the gradient of the tangent and a point on it is known, the equation can be found by using: *y* – *y*1 = *m*(*x* –*x*1), where (*x*1; *y*1) = (-1; 0).

*y* – *y*1 = *m*(*x* –*x*1)

* *y* – 0 = 1(*x* + 1)
* *y* = *x* + 1 is the equation of tangent FG at A(-1; 0) √√

**Summary Assessment**

1. Determine the equation of the tangent to the given circleat the given point:

(a) x 2 + y 2 = 29 at (-5;2)

(b) (x - 3)2 + ( y + 4)2 = 63 at (-1; 3)

1. A circle has centre (2;-1) and touches the x-axis at exactly one point. The circle is

called **tangent** to the axis.

* 1. Draw a picture to illustrate the above description.
  2. What are the co-ordinates of the point of tangency?
  3. Determine the equation for the circle.

1. 

The circle x2 + y2  + - 4 x - 6y -16 = 0 is given in the diagram alongside. The point P(a;1) lies in the circle in the second quadrant.

3.1. Find the value of a.

3.2. Determine the coordinates of the centre of the circle.

3.3 Determine the equation of the tangent to the circle at P in the form

ax + by + c = 0.

3.4. Find the equation of the diameter of the given circle, parallel to the tangent at

P.

3.5. Show that M, the y-intercept of the circle lies on this diameter.

3.6. Calculate the size of θ, the angle which the diameter makes with the positive

y-axis. (Give your answer rounded off to 2 decimal places).

#### Guided reflection on Summary Assessment

|  |
| --- |
| 1. Use the answers given for the summary assessment to check if your answers were correct. 2. Did you realize why you got some your answers incorrect? 3. Have you been able to correct errors you made? |

**Answers for summary assessment**

1.



2. (a)



1. (2; 0)
2. *r* = 1

Equation of circle: (*x* -2)2 + (*y* + 1)2 = 1

(d) A = π***r*2**

= π**(1 unit)2**

**≈ 3,14 square units**

* 1. **Let (*x*; *y*) = (a; 1)**

**Substitute *x* = a and *y* = 1 into the circle equation:** *x*2 + *y*2  - 4*x* - 6*y* -16 = 0

*a*2 + (1)2  - 4*a* – 6(1) -16 = 0

*a*2 + 1 - 4*a* – 6 -16 = 0

*a*2 - 4*a* – 21 = 0

(*a* + 3)(*a* - 7) = 0

*a* = -3 or *a* = 7

**Reject *a* = 7.**

***a* = -3**

* 1. *x*2 + *y*2  - 4*x* - 6*y* -16 = 0

*x*2 – 4*x* + *y*2  - 6*y* = 16

(*x*2 – 4*x* + 4) + (*y*2  - 6*y* + 9) = 16 + 4 + 9

(*x* -2)2 + (*y* - 3)2 = 29.

The coordinates of the centre are N(2; 3) √√

3.3. Let (*x*1; *y*1) = (-3; 1) and (*x*2; *y*2) = (2; 3)

*m*NP =

= 3-1\_\_

2- (-3)

=

The tangent is perpendicular to the radius, therefore find the gradient of the tangent, using the fact that the product of the gradients of perpendicular lines is –1.

*m*NP × *m*tangent = -1

* × *m*tangent = -1
* *m*tangent =

Now that the gradient of the tangent and a point on it is known, the equation of the tangent at P can be found by using: *y* – *y*1 = *m(x* –*x*1), where (*x*1; *y*1) = (-3; 1).

*y* – *y*1 = *m*(*x* –*x*1)

* *y* = –

**3.4**  mMN = mtangent  **(gradient of parallel lines are equal)**

mMN =

Now that the gradient of the diameter, MN, and a point on it, are known, *y*ou can find the equation of MN, by using: *y* – *y*1 = m (*x* –*x*1), where (*x*1; *y*1) = (2; 3).

*y* – *y*1 = *m(x* –*x*1)

* is the equation of diameter.
  1. Let *x* = 0 in *x*2 + *y*2  - 4*x* - 6*y* -16 = 0,

Then *y*2  - 6*y* -16 = 0

**(*y*-8)(*y* + 2) = 0**

***y* = 8 or *y* = -2**

**(reject *y* = -2 because the *y*-coordinate of M is positive)**

∴**coordinates of M are (0; 8)**

**Consider the equation of the diameter,**

**LHS = 8**

**RHS = -5/2(0) + 8 = 8**

∴ **M(0; 8) lies on the diameter.**



∴*tan* OX =

∴OX = 111,80º

In ∆ORM,

*θ* + 90º = 111,80º (ext. < of ∆ORM)

∴ *θ* = 21,80º